DEFINITION A nonempty set of elements G is said to form a *group* if in G there is defined a binary operation, called the product and denoted by $\cdot$, such that

1. $a, b \in G$ implies that $a \cdot b \in G$ (closed).
2. $a, b, c \in G$ implies that $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (associative law).
3. There exists an element $e \in G$ such that $a \cdot e = e \cdot a = a$ for all $a \in G$ (the existence of an identity element in G).
4. For every $a \in G$ there exists an element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$ (the existence of inverses in G).

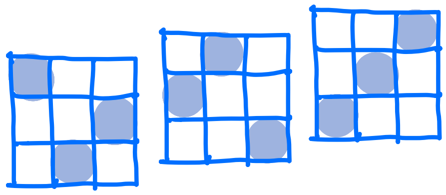
Group Theory

Herstein - Topics in Algebra
Definition of a group

1873 Lie - Lie groups

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

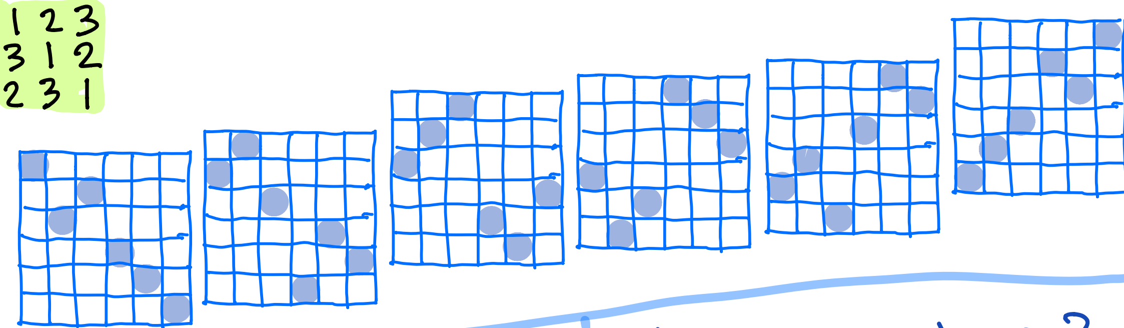
0	1	2
1	2	0
2	0	1



*	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	1	6	4	5
3	3	1	2	5	6	4
4	4	5	6	1	2	3
5	5	6	4	3	1	2
6	6	4	5	2	3	1

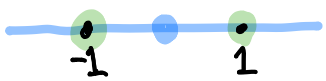
1	2	3	4	5	6
2	3	1	6	4	5
3	1	2	5	6	4
4	5	6	1	2	3
5	6	4	3	1	2
6	4	5	2	3	1

What is associative law?



Lie Group spheres?
S^0, S^1, S^3 only

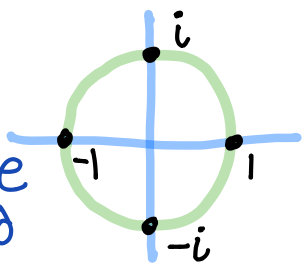
$\mathbb{R}$
ordered



*	1	-1
1	1	-1
-1	-1	1

sphere S^0

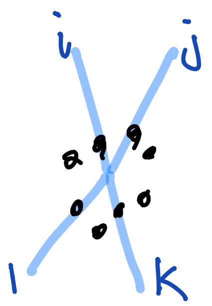
$\mathbb{C}$
commutative
not ordered



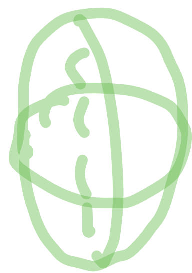
*	1	i	-1	-i
1	1	i	-1	-i
i	i	-1	-i	1
-1	-1	-i	1	i
-i	-i	1	i	-1

sphere S^1 in $\mathbb{R}^2$

$\mathbb{Q}$uat
associative
not commutative



*	1	i	j	k	-1	-i	-j	-k
1	1	i	j	k	-1	-i	-j	-k
i	i	-1	k	-j	-i	1	-k	j
j	j	-k	-1	i	-j	k	1	-i
k	k	j	-i	-1	-k	-j	i	1
-1	-1	-i	-j	-k	1	i	j	k
-i	-i	1	-k	j	i	-1	k	-j
-j	-j	k	1	-i	j	-k	-1	i
-k	-k	-j	i	1	k	j	-i	-1



sphere S^3 in $\mathbb{R}^4$

$\mathbb{O}$ Octonians 16x16 table $\boxed{\text{not associative}}$

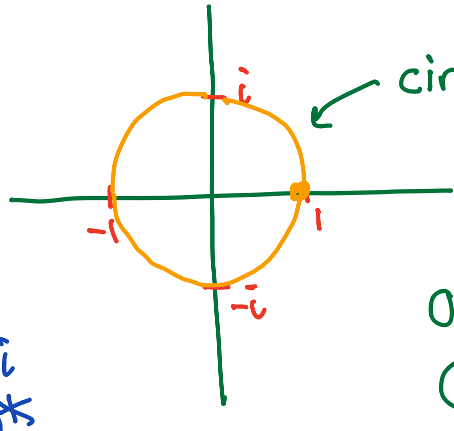
Seminar notes

Complex numbers

$\mathbb{C} \approx \mathbb{R}^2$ with multiplication $\quad (a+bi)(c+di) = (ac-bd) + (ad+bc)i$

$a+bi \quad (a,b)$

$i = \sqrt{-1}, \quad i^2 = -1$



circle S^1 forms a group under multiplication

(Cabbie story)

$(a+bi)^* = a-bi$

$(a+bi)(a+bi)^*$

$= (a+bi)(a-bi)$

$= a^2+b^2 = |(a,b)|^2$

Or in polar coords

$(r,\theta)(s,\gamma) = (rs, \theta+\gamma)$

lengths multiply
angles add

$$\boxed{re^{\theta i} se^{\gamma i} = rs\, e^{(\theta+\gamma)i}}$$

alternate notation

Euler's formula $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ $\quad e^{i\theta}$ rotates ↶ by θ

$$e^{i(a+b)} = e^{ia} e^{ib}$$

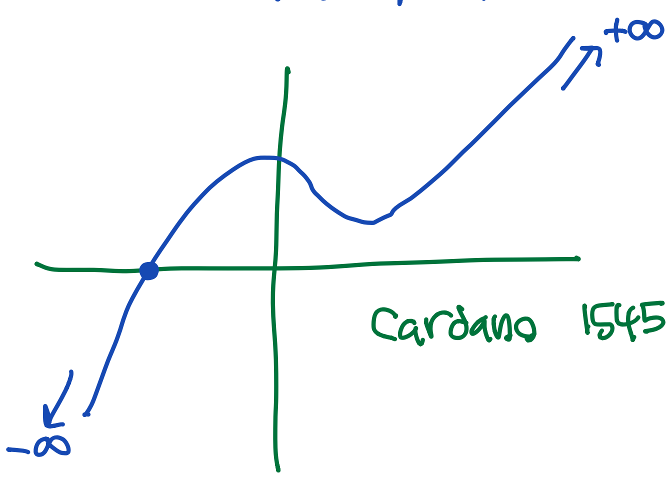
$$= (\cos a + i\sin a)(\cos b + i\sin b)$$

$$\boxed{\cos(a+b)} + i\boxed{\sin(a+b)} = \boxed{(\cos a\cos b - \sin a\sin b)} + i\boxed{(\cos a\sin b + \sin a\cos b)}$$

Historical significance: Over $\mathbb{C}$, every poly $f(x)$ factors into linear terms.

$$x^2+1 = (x+i)(x-i)$$

$$f(x) = x^3+ax^2+bx+c = 0$$



Cardano 1545

Formula for real root needs $\mathbb{C}$.

Many at time: Ugh!

Quaternions

$\mathbb{Q} = \mathbb{C}^2 = \mathbb{R}^4$ with multiplication

Quaternions

$a+bi+cj+dk$

$a + v$

$v \in \mathbb{R}^3$

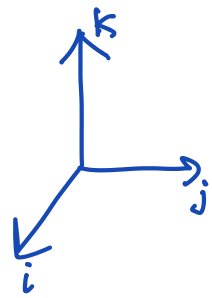
$v = bi+cj+dk$

$i^2=j^2=k^2=-1$

$ij=-ji=k$

$jk=-kj=i$

$ki=-ik=j$



right hand rule

cross product

$$(\underbrace{a+bi+cj+dk}_{v})(w+xi+yj+zk) = \dots$$

use above

$$(a+v)(b+w) = \underbrace{(ab - v\cdot w)}_{\text{scalar}} + \underbrace{(aw+bv+v\times w)}_{\text{vector}}$$

not commutative

$$(a+v)^* = a-v$$

$$(a+v)(a-v) = (aa+v\cdot v) + (\cancel{-av+av}^{\,0} + \cancel{v\times v}^{\,0}) = \text{length}^2$$

$$(b+w) \qquad (ab-v\cdot w)+(aw+bv+v\times w)$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \Longrightarrow \boxed{\cos\theta + v\sin\theta}$$

Euler's formula

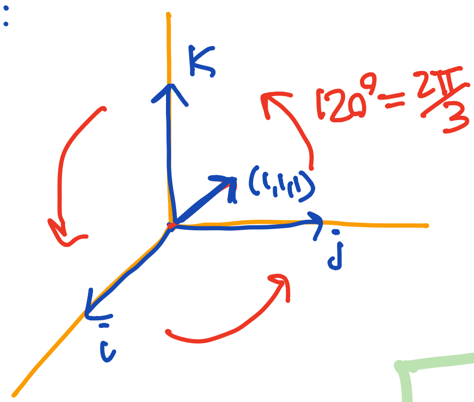
unit complex number

unit quaternion, where $|v|=1$

Let $q = \cos\theta/2 + v\sin\theta/2$ $\quad w = bi+cj+dk$,

Then $\boxed{w \mapsto q w q^{-1} \text{ rotates by } \theta \text{ around axis } v \text{ in } \mathbb{R}^3}$

Example:



Rotate $2\pi/3$ around axis $(1,1,1)$

$v = (1,1,1)/\sqrt{3}$

$q = \cos(\pi/3) + v\sin(\pi/3)$

$= \frac{1}{2} + \cancel{\frac{\sqrt{3}}{2}}\,(1,1,1)/\cancel{\sqrt{3}}$

$$\boxed{\begin{aligned} q &= \frac{1+i+j+k}{2} \\ q^* = q^{-1} &= \frac{1-i-j-k}{2} \end{aligned}}$$

("Second Life" story)

$$i^2=j^2=k^2=-1$$
$$ij=-ji=k$$
$$jk=-kj=i$$
$$ki=-ik=j$$

$$\Rightarrow \quad \begin{array}{c|cccc} & 1 & i & j & k \\ \hline 1 & 1 & i & j & k \\ i & i & -1 & k & -j \\ j & j & -k & -1 & i \\ k & k & j & -i & -1 \end{array}$$

(get ready to multiply)

$$\boxed{qwq^{-1} = \left(\frac{1+i+j+k}{2}\right)(bi+\underbrace{cj+dk}_{\text{expand}})\left(\frac{1-i-j-k}{2}\right)}$$

$$\left(\frac{1+i+j+k}{2}\right) \mathbf{i} \left(\frac{1-i-j-k}{2}\right) = \frac{1}{4}(1+i+j+k)(i+1-k+j)$$

$$\begin{array}{c|cccc} & 1 & i & j & k \\ \hline 1 & 1 & i & j & k \\ i & i & -1 & k & -j \\ j & j & -k & -1 & i \\ k & k & j & -i & -1 \end{array} \quad = \mathbf{j}$$

$$\left(\frac{1+i+j+k}{2}\right) \mathbf{j} \left(\frac{1-i-j-k}{2}\right) = \frac{1}{4}(1+i+j+k)(j+k+1-i)$$

$$\begin{array}{c|cccc} & 1 & i & j & k \\ \hline 1 & 1 & i & j & k \\ i & i & -1 & k & -j \\ j & j & -k & -1 & i \\ k & k & j & -i & -1 \end{array} \quad = \mathbf{k}$$

$$\left(\frac{1+i+j+k}{2}\right) \mathbf{k} \left(\frac{1-i-j-k}{2}\right) = \frac{1}{4}(1+i+j+k)(k-j+i+1)$$

$$\begin{array}{c|cccc} & 1 & i & j & k \\ \hline 1 & 1 & i & j & k \\ i & i & -1 & k & -j \\ j & j & -k & -1 & i \\ k & k & j & -i & -1 \end{array} \quad = \mathbf{i}$$

Cayley–Dickson construction.

x^* is "conjugate" of x

$x \times x^*$ = length squared (real, positive)

$\mathbb{R}$: $a^* = a$, $aa^* = a^2$ ✓

$\mathbb{C}$ $(a+bi)^* = (a-bi)$, $(a+bi)(a-bi) = a^2+b^2 = |(a,b)|^2$ ✓

$\mathbb{Q}$ Quaternions $(a+bi+cj+dk)(a-bi-cj-dk) = a^2+b^2+c^2+d^2$
$= |(a,b,c,d)|^2$ ✓

	a	$-bi$	$-cj$	$-dk$
a	a^2	$-$	$-$	$-$
bi		b^2	$-$	$-$
cj			c^2	$-$
dk				d^2

General case: $(a,b) \times (c,d) = (ac - d^*b,\ da + bc^*)$

$(a,b)^* = (a^*, -b)$

$$(a,b) \times (a,b)^* = (a,b) \times (\underset{c}{a^*}, \underset{d}{-b})$$

$(ac - d^*b,\ da + bc^*)$

$$= (aa^* + b^*b,\ \underbrace{\cancel{-ba + ba}}_{0})$$ ✓

$\mathbb{O}$ octonions $\approx \mathbb{R}^8$ + multiplication

$\mathbb{S}$ sedonians $\approx \mathbb{R}^{16}$ + multiplication

⋮

Twilight Zone

⋮

1545 Cardano complex numbers (for cubic equations)

1832 Galois Groups, Fields, symmetry

1843 Hamilton Quaternions

1843 Graves Octonions (Cayley also)

1919 Dickson Cayley–Dickson construction