Combinatorics Jan 20, 2022 #2

$\boxed{\text{Binomial Coefficients}}$

$\binom{n}{k}$ = # ways to choose k elements from n possibilities

$$= \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 2\cdot 1}$$

Pascal's triangle

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
```

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$3 + 3 = 6$

$\binom{3}{1} + \binom{3}{2} = \binom{4}{2}$

$\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$

$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

use first element? don't

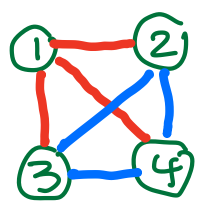
$\{1, \ldots, n\}$ $\{1, \ldots\}$ $\{\ldots\}$

1	2	3	4	
x	x			12
x		x		13
x			x	14
	x	x		23
	x		x	24
		x	x	34

$\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$

6 = 3 + 3

12, 13, 14 | 23, 24, 34




```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1      16
  1   5  10  10   5   1
```

$(x+y)^0 = 1$

$(x+y)^1 = x+y$

$(x+y)^2 = x^2+2xy+y^2$

$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$

```
      1
    1   1
  1   2   1
1   3   3   1
```

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{n}y^n$$

$$= (x+y)(x+y)^{n-1}$$

$$(x+y)^{n-1} = \binom{n-1}{0}x^{n-1} + \binom{n-1}{1}x^{n-2}y + \binom{n-1}{2}x^{n-3}y^2 + \dots$$

$$x\left[\binom{n-1}{0}x^{n-1} + \binom{n-1}{1}x^{n-2}y + \binom{n-1}{2}x^{n-3}y^2 + \dots\right]$$

$$+y\left[\binom{n-1}{0}x^{n-1} + \binom{n-1}{1}x^{n-2}y + \binom{n-1}{2}x^{n-3}y^2 + \dots\right.$$

$$1\ x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$x=1\quad y=1$

$(1+1)^4 = 2^4 = 16$

$$(x+y)(x+y)(x+y)(x+y)$$

x x x x
x x x y
x x y x
x x y y

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$1 = 1$
$1+2 = 3$
$1+2+3 = 6$
$1+2+3+4 = 10$

1 2 3 4 5 6 $\quad 3 \cdot 7 = 21$

even $\quad 1 \quad 2 \quad \dots \quad n-1 \quad n$

$$\frac{n}{2}(n+1) \qquad \frac{(n+1)n}{2\cdot 1} = \binom{n+1}{2}$$

1 2 3 4 5 6 7

$$\frac{n-1}{2}(n+1)+\frac{1}{2}(n+1)=\frac{n}{2}(n+1)$$

picture for $1+2+\dots+n$

$n=4$

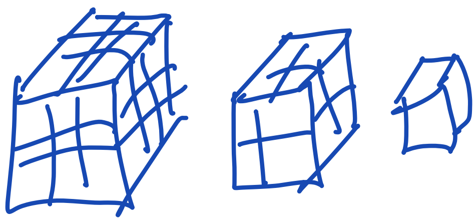
$$\frac{n(n+1)}{2}=\binom{n+1}{2}$$

$$f(1)=1+2+\dots+n=\binom{n+1}{2}$$

$$f(2)=1^2+2^2+\dots+n^2=\sim$$

$$f(3)=1^3+2^3+\dots+n^3=\sim$$

$$f(3)=f(1)^2$$



$$\frac{d}{dx}f(x)=\lim_{\varepsilon\to 0}\frac{f(x+\varepsilon)-f(x)}{\varepsilon}$$

discrete $\quad \Delta f(n) \underset{(\varepsilon=1)}{=} f(n+1)-f(n)$

$e^x \quad x^n \quad x^{n-1} \quad \dots$

2^n

$2^{n+1}-2^n=2^n$

$$\Delta\binom{n}{k}=\binom{n+1}{k}-\binom{n}{k}=\binom{n}{k-1}$$

$$\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}$$

$$\binom{n+1}{k}=\binom{n}{k-1}+\binom{n}{k}$$

$$\frac{\partial}{\partial x}x^n=nx^{n-1}$$

$$\Delta\binom{n}{k}=\binom{n}{k-1}$$

what is $f(n) = 1+2+\dots+n$?

$$\Delta f(n) = f(n+1) - f(n) = n+1$$
$$= \binom{n}{1} + \binom{n}{0}$$

$f(2) = 1+2 = 3$

$$\Rightarrow f(n) = \binom{n}{2} + \binom{n}{1} + C$$
$$f(2) = \underbrace{\binom{2}{2}}_{1} + \underbrace{\binom{2}{1}}_{2} + 0$$

$$\boxed{\binom{n}{2} + \binom{n}{1} = \binom{n+1}{2}}$$

$$\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$$

n-dim symmetric matrices

$$\frac{n(n+1)}{2} \qquad \frac{3\cdot 4}{2} = \frac{12}{2} = 6$$

$n=3$

6 abcdef

$$3\ \underbrace{\begin{bmatrix} a & a & d & e \\ d & b & b & f \\ e & f & c & c \end{bmatrix}}_{4} \div 2$$

How many monomials of deg d in n vars?

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

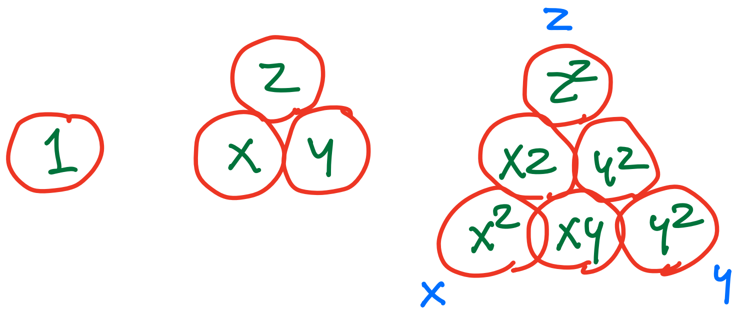
$n=1$	1	x	x^2	x^3	x^4	x^5
	1	1	1	1	1	1
$n=2$	1	x, y	x^2, xy, y^2	x^3, x^2y, xy^2, y^3		
	1	2	3	4		
$n=3$	1	x, y, z	$x^2, y^2, z^2, xy, xz, yz$			
	1	3	6			

$1 = 1$

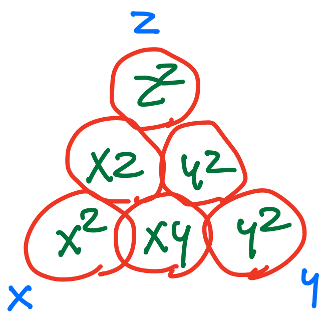
$1+2 = 3$

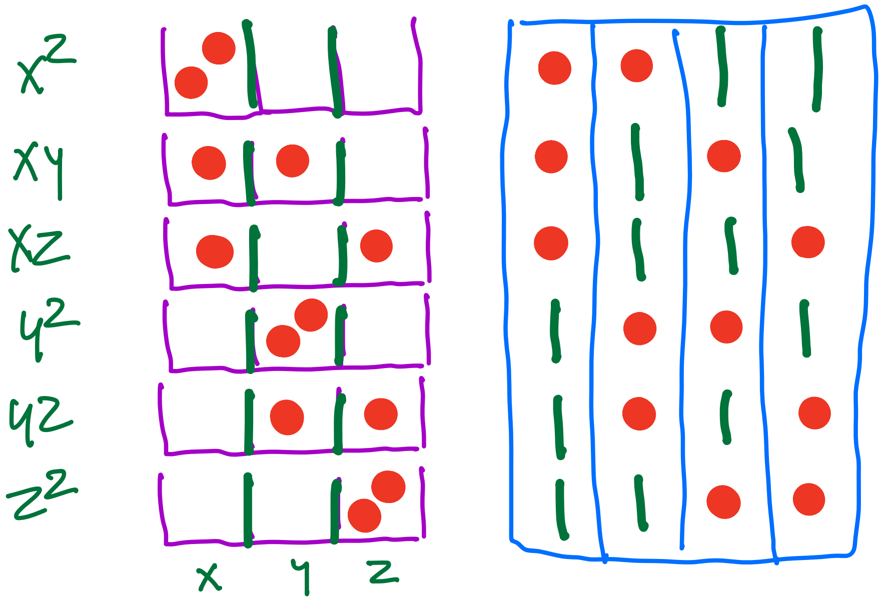
$1+2+3 = 6$

$1+2+3+4 = 10$



$\boxed{\text{Bars \& Stars argument}}$



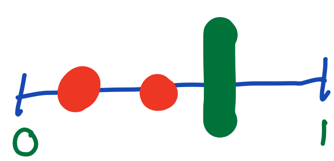


n vars deg d $\quad \binom{n-1+d}{n-1} = \binom{n-1+d}{d}$

$n=3$, d varies $\quad \binom{d+2}{2} \quad \binom{n+1}{2}$

1, 3, 6, 10

Bose Einstein statistics



A, B |

AB|	1/4	1/3	••|
A|B B|A	1/2	1/3	•|•
|AB	1/4	1/3	••|