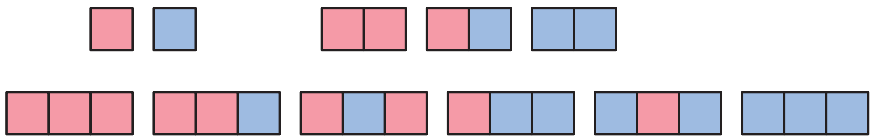
Exam 2

Combinatorics, Dave Bayer, April 6-10, 2022

Please show all of your work. You will be graded for both your answers and your explanations. You need not complete the entire exam; the questions vary in difficulty.

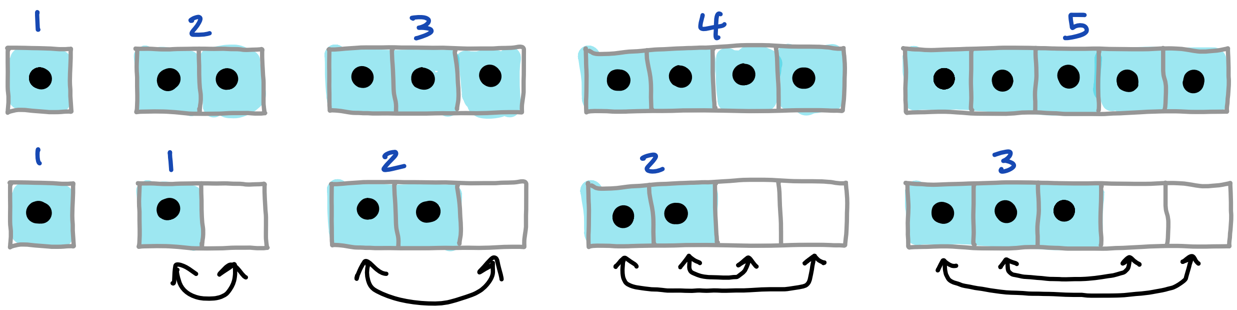
[1] How many ways can we color the cells of a strip of n squares using at most k colors, counting two patterns as the same if one is a reversal of the other?



$G = \{\mathrm{Id}, \leftrightarrow\}$ $\quad |G| = 2$

Id $\quad k^n$

$\leftrightarrow \quad k^{\lceil \frac{n}{2} \rceil}$

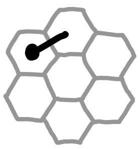


$$\boxed{\left(k^n + k^{\lceil \frac{n}{2} \rceil}\right)/2}$$

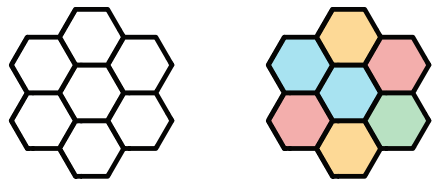
k	n	k^n	$k^{\lceil \frac{n}{2} \rceil}$	$(k^n + k^{\lceil \frac{n}{2} \rceil})/2$
2	1	2	2	2
2	2	4	2	3
2	3	8	4	6

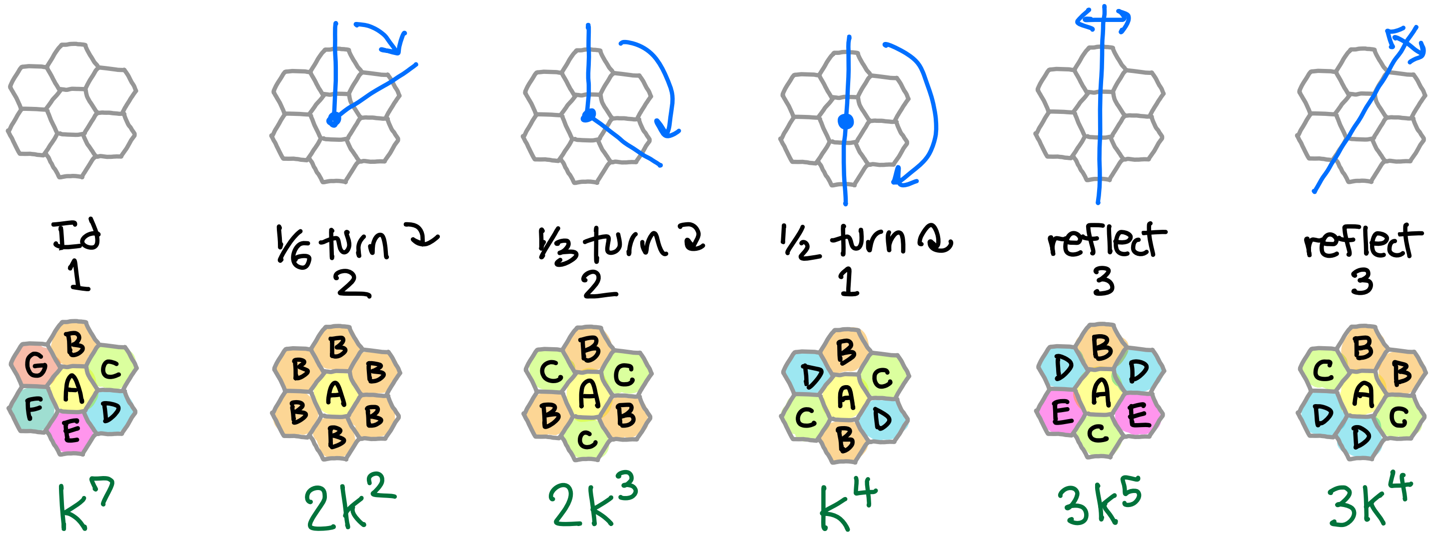
☑

[2] How many ways can we color the cells of this beehive using at most k colors, up to the dihedral group of rotations and flips? Confirm your answer for $k = 2$, by finding all patterns up to symmetry.



$|G| = 6 \cdot 2 = 12$



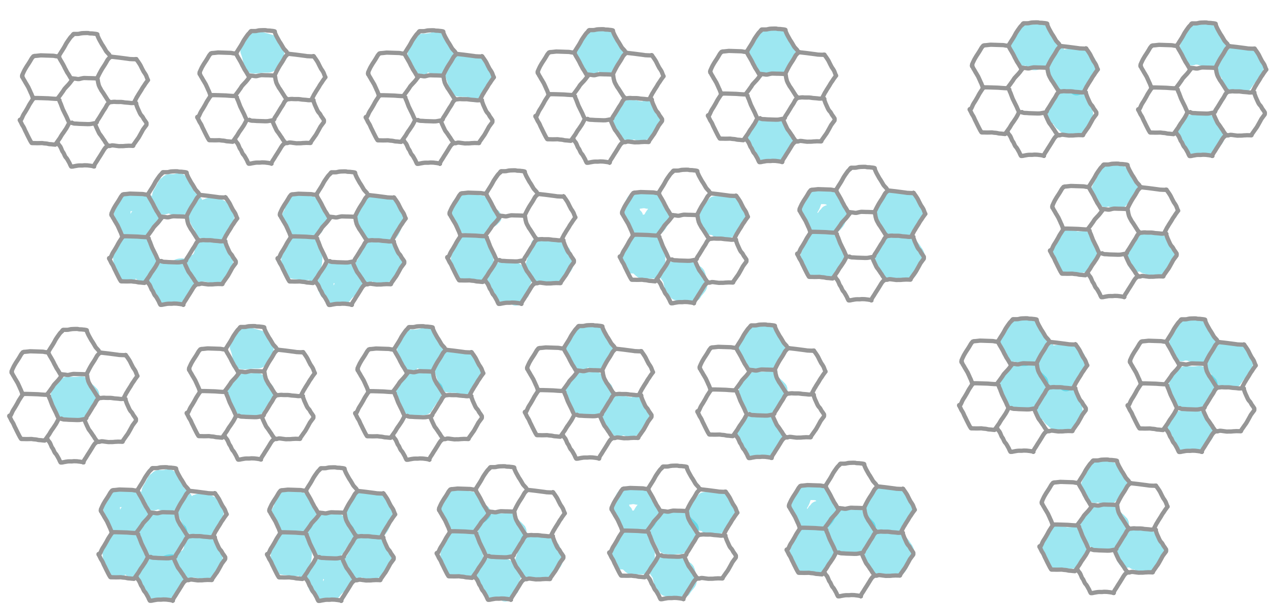


$$\boxed{(2k^2 + 2k^3 + 4k^4 + 3k^5 + k^7)/12}$$

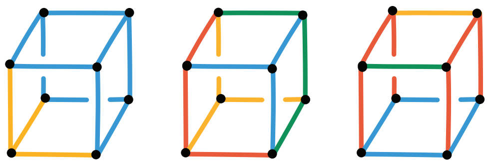
	2	3	4	5	7		12	
	2	2	4	3	1	total	count	/k
1	2	2	4	3	1	12	1	1
2	8	16	64	96	128	312	26	13
3	18	54	324	729	2,187	3,312	276	92
4	32	128	1,024	3,072	16,384	20,640	1,720	430
5	50	250	2,500	9,375	78,125	90,300	7,525	1,505

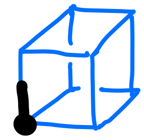
https://oeis.org/A027670

check $k=2$ $\boxed{26}$ ✓



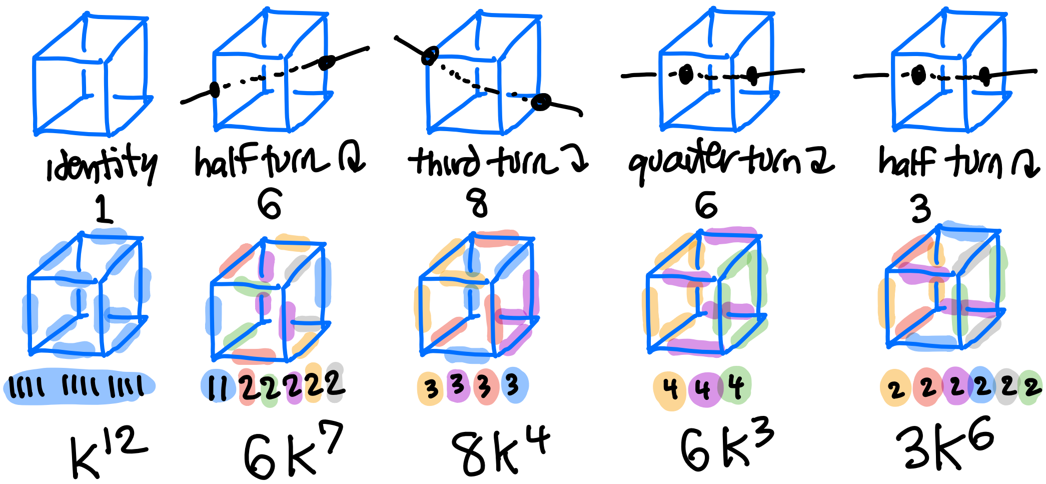
[3] How many ways can we color the edges of a cube using at most k colors, up to the group of rotational symmetries? Can you check your answer for k = 2?





$|G| = 8 \cdot 3 = 24$

$1+6+8+6+3 = 24$ ✓

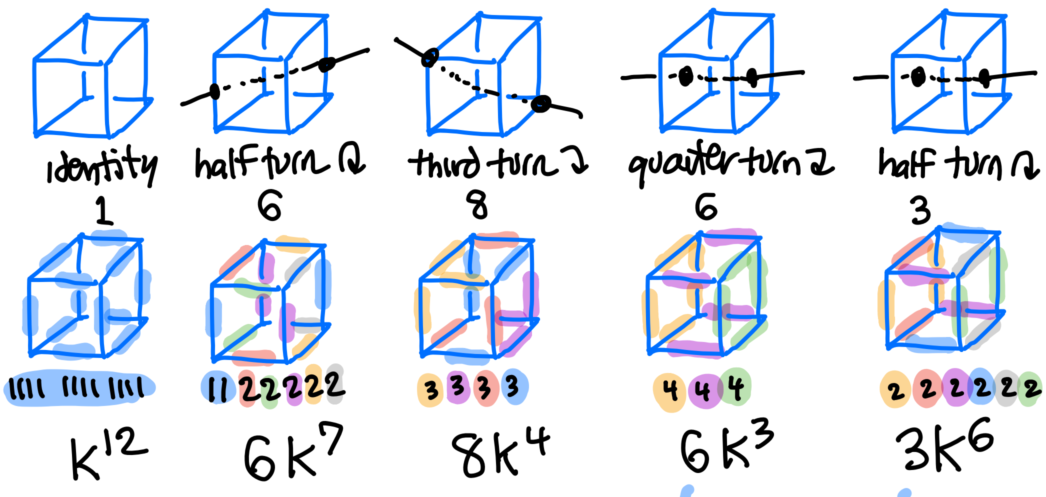


$$\boxed{(6k^3 + 8k^4 + 3k^6 + 6k^7 + k^{12})/24}$$

	3	4	6	7	12		24
	6	8	3	6	1	total	count
1	6	8	3	6	1	24	1
2	48	128	192	768	4,096	5,232	218
3	162	648	2,187	13,122	531,441	547,560	22,815

https://oeis.org/A060530

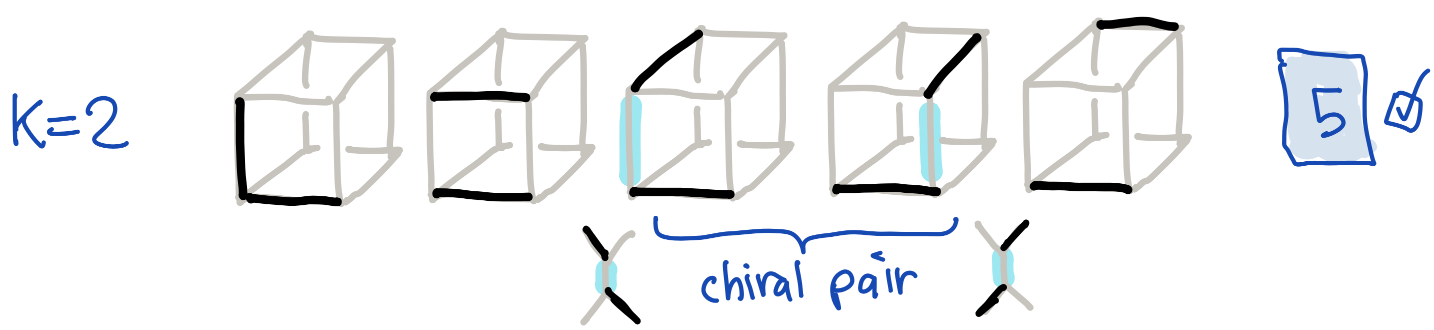
One way to check k=2 is to count subsets of each size, up to symmetry. We can confirm the smaller counts by hand, and see they add up.



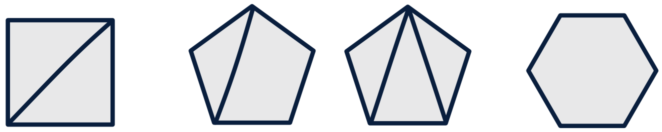
0	1	1			1	1	1
1	12		2				
2	66	5		1			6
3	220		10		4		
4	495	10		5		3	15
5	792		20				
6	924	10		10	6		20
7	792		20				
8	495	5		10		3	15
9	220		10		4		
10	66	1		5			6
11	12		2				
12	1			1	1	1	1
	4,096	**32**	**64**	**32**	**16**	**8**	**64**

1	6	8	6	3
1	6	8	6	3
12	12	0	0	0
66	36	0	0	18
220	60	32	0	0
495	90	0	18	45
792	120	0	0	0
924	120	48	0	60
792	120	0	0	0
495	90	0	18	45
220	60	32	0	0
66	36	0	0	18
12	12	0	0	0
1	6	8	6	3
4,096	**768**	**128**	**48**	**192**

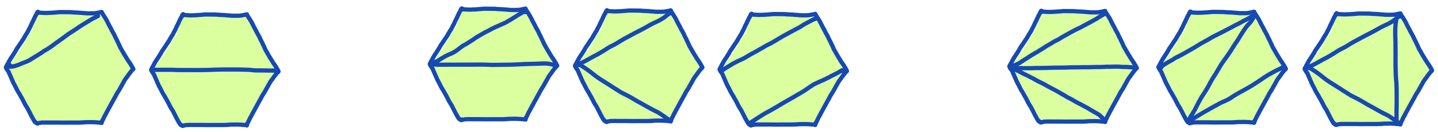
Total	Count
24	1
24	1
120	5
312	13
648	27
912	38
1,152	48
912	38
648	27
312	13
120	5
24	1
24	1
5,232	**218**



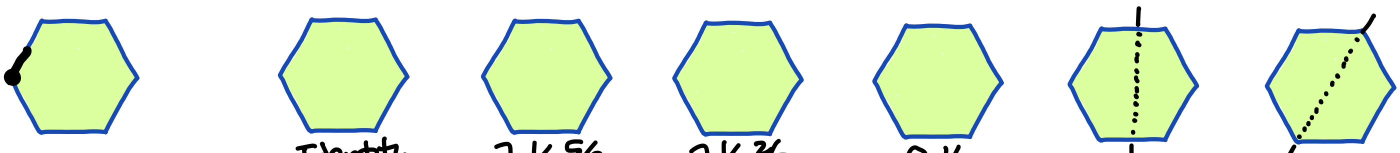
[4] Let $f(n)$ be the number of ways of dissecting an n-gon by at least one cut, up to the dihedral group of rotations and flips. As shown, $f(4) = 1$ and $f(5) = 2$. Find $f(6)$ two ways, by drawing the cases by hand and by using Burnside's lemma.



$f(6)=8$



$6 + 3 = 9$ ☑ $12 + 6 + 3 = 21$ ☑ $6 + 6 + 2 = 14$ ☑



$|G|=12$

Identity 1

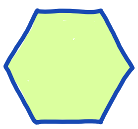
↻ 1/6, 5/6 2

↻ 1/3, 2/3 2

↻ 1/2 1

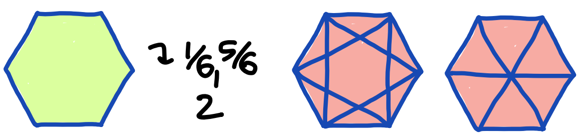
3

3



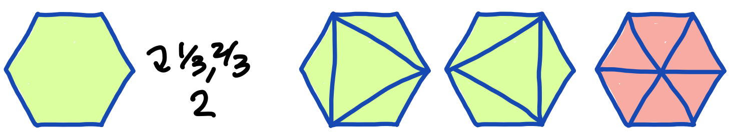
Identity 1

$9+21+14 = 44$



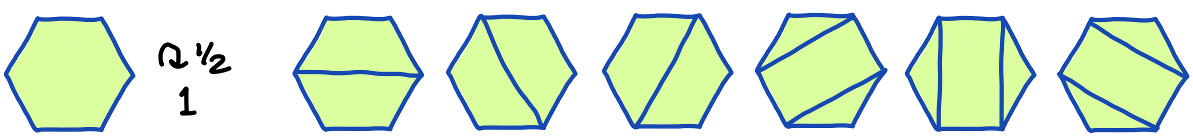
↻ 1/6, 5/6 2

0



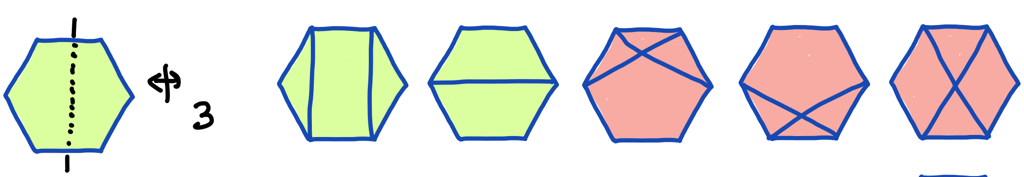
↻ 1/3, 2/3 2

2



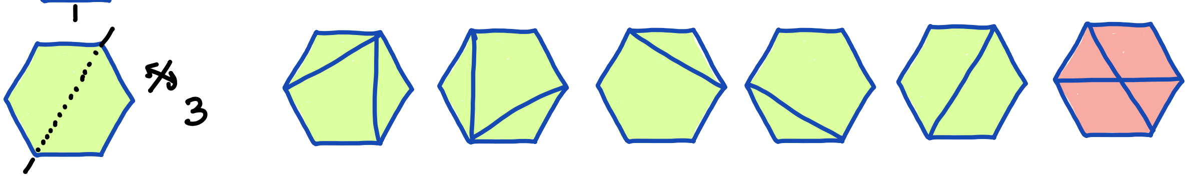
↻ 1/2 1

6+6 pairs 12



3

2

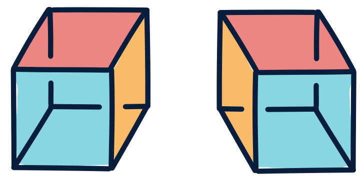


3

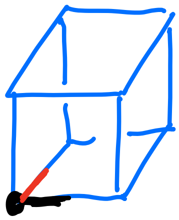
5+5 pairs 10

$(44+2\cdot 2+12+3\cdot 2+3\cdot 10)/12 = 96/12 = 8$ ☑

[5] How many ways can we color the faces of a cube using at most k colors, up to the group of symmetries generated by rotations and reflections ("look in the mirror")?



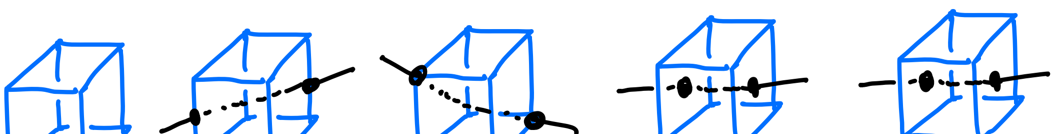
There are 48 symmetries of the cube, including reflections.



① pick a corner (8)
② pick an edge meeting that corner (3)
③ pick a rotational direction (orientation) (2)

$8 \cdot 3 \cdot 2 = \boxed{48}$ we have studied the 24 rotations that preserve orientation.

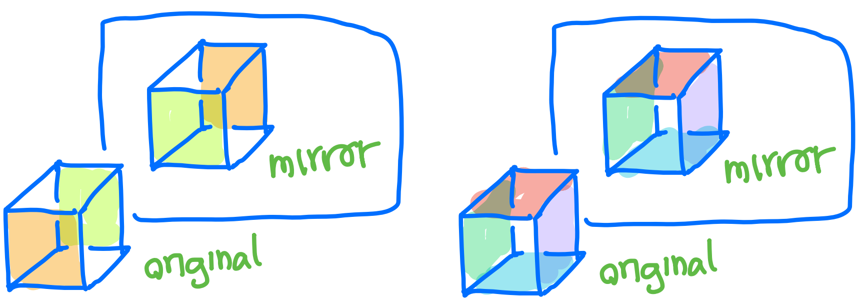
It is harder to classify the 24 symmetries that reverse orientation: Some involve not one but 3 reflections!



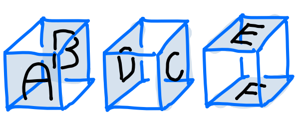
identity 1 | half turn 6 | third turn 8 | quarter turn 6 | half turn 3

$1+6+8+6+3=24$ ✓

For each rotation, we will also group faces by what happens in the mirror.



mirror swaps front and back faces, leaves sides alone

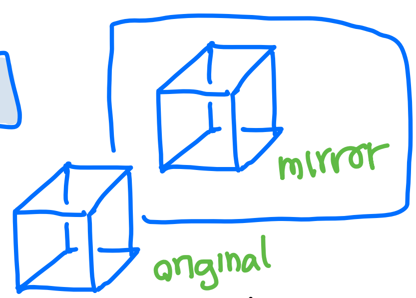


mirror swaps A and B

rotations

k^6

ⒶⒷⒸⒹⒺⒻ



identity
1

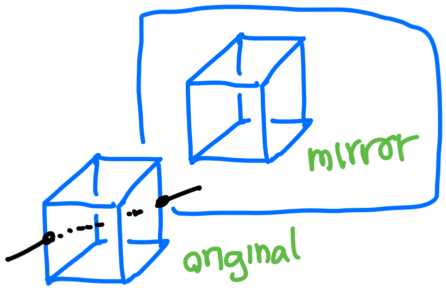
reflections

k^5

(AB) Ⓒ Ⓓ Ⓔ Ⓕ

rotations

$6k^3$

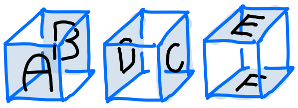


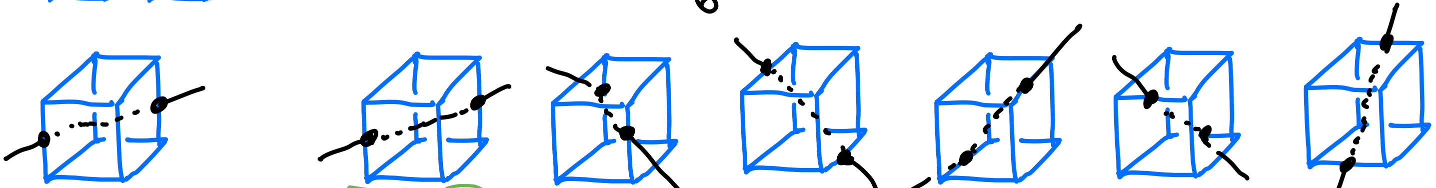
half turn

6

reflections

$2k^4+4k^2$





AD BC EF | ADBC EF | ACBD EF | A B CF DE | A B CD EF | AEBF CD | AFBE CD

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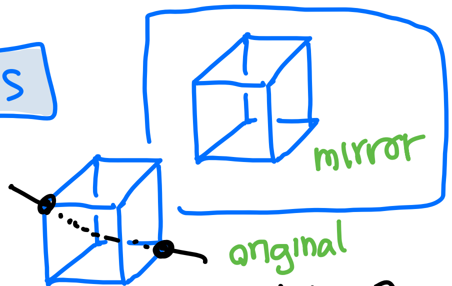
rotations

$8k^2$



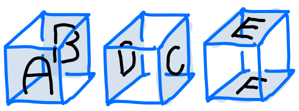
third turn

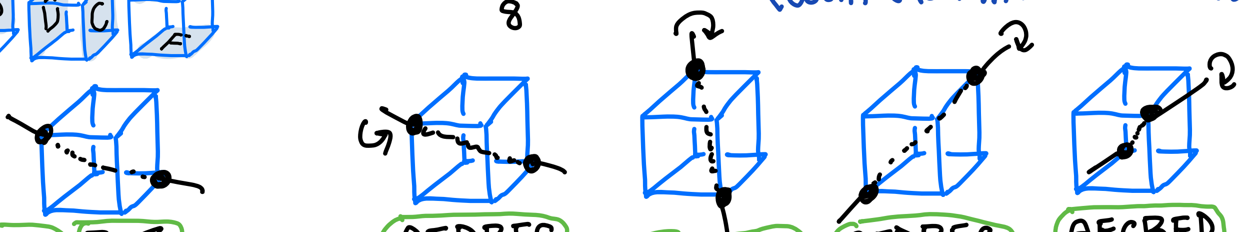
8

reflections

$8k$

(count each axis either way)





AED BCF | AEDBFC | ACFBDE | AFDBEC | AECBFD

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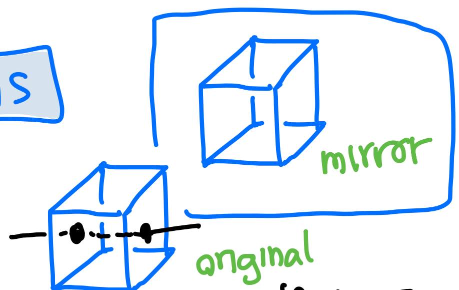
rotations

$6k^3$



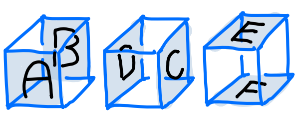
quarter turn

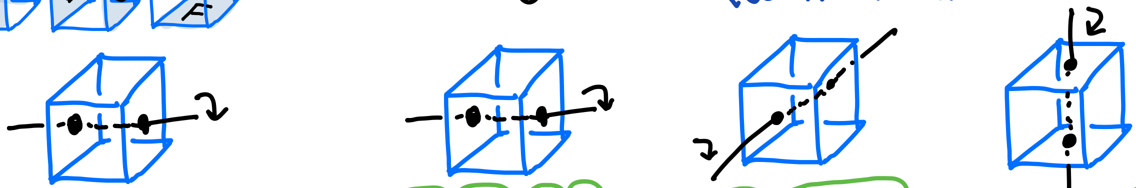
6

reflections

$4k^4+2k^2$

(count each axis either way)

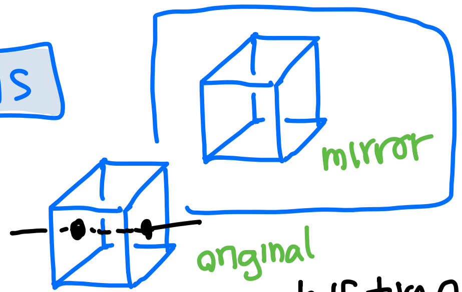




AEBF C D | AE BF C D | AB CFDE | AD BC E F

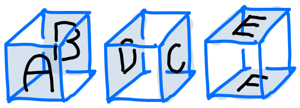
rotations

$3k^4$



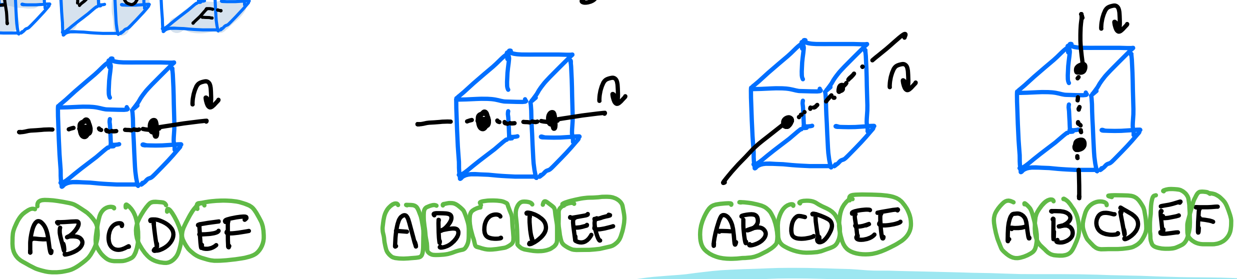
reflections

$2k^5 + k^3$



half turn

3



rotations $k^6 + 6k^3 + 8k^2 + 6k^3 + 3k^4$

reflections $k^5 + 2k^4 + 4k^2 + 8k + 4k^4 + 2k^2 + 2k^5 + k^3$

$$\boxed{(8k + 14k^2 + 13k^3 + 9k^4 + 3k^5 + k^6)/48}$$

	1	2	3	4	5	6		48
	8	14	13	9	3	1	total	count
1	8	14	13	9	3	1	48	1
2	16	56	104	144	96	64	480	10
3	24	126	351	729	729	729	2,688	56
4	32	224	832	2,304	3,072	4,096	10,560	220
5	40	350	1,625	5,625	9,375	15,625	32,640	680

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