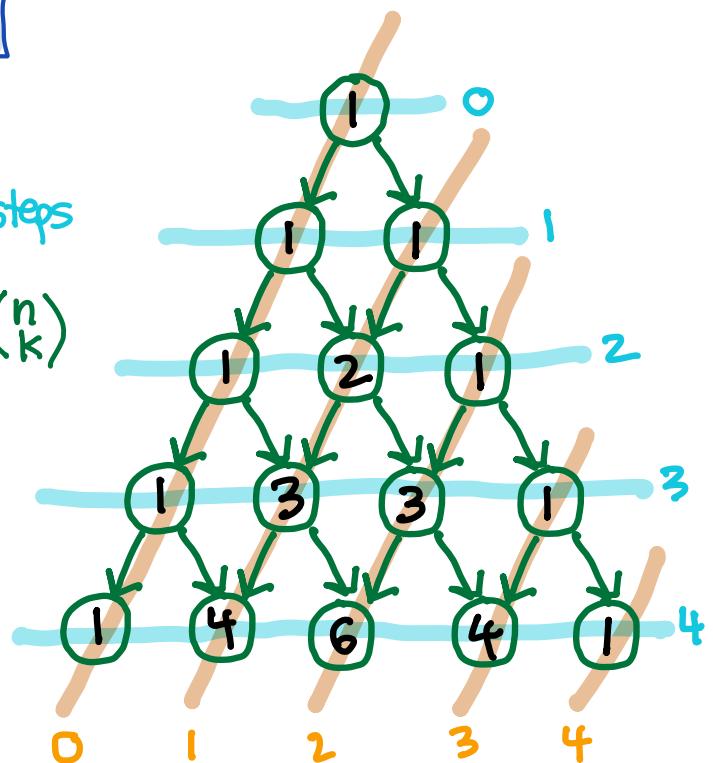
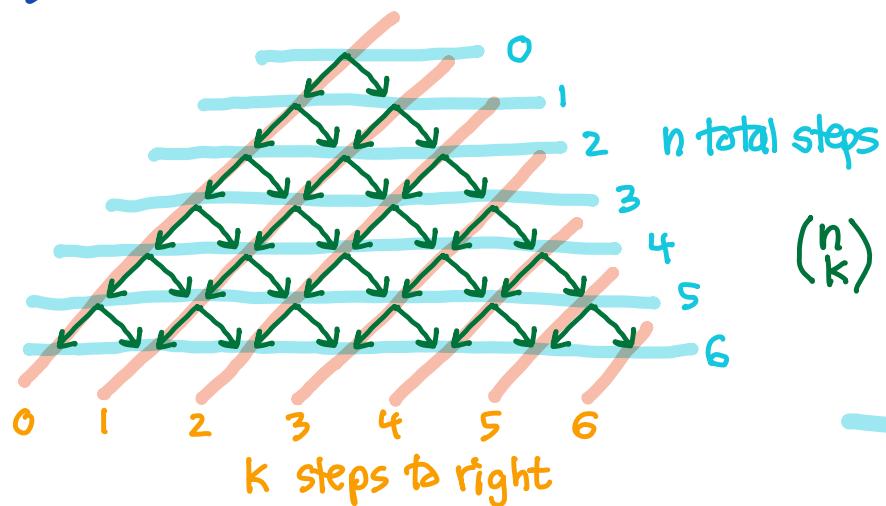


Combinatorics #3, January 25, 2022



$(n-1 \choose k-1)$ $(n-1 \choose k)$ 1 3 4

Each path comes from left or right

$$(n \choose k) = (n-1 \choose k-1) + (n-1 \choose k)$$

1	4	10	20	end
1	3	6	10	
1	2	3	4	
1	1	1	1	

start

$$\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$$

(all over)

→ → → ↑↑↑

Inclusion-exclusion

1	5	9	17	34
1	4	4	8	17
1	3	0	4	9
1	2	3	4	5
1	1	1	1	1

avoid

=

1	5	15	35	70
1	4	10	20	35
1	3	6	10	15
1	2	3	4	5
1	1	1	1	1

free-for-all

		6	18	36
		6	12	18
1	3	6	6	6
1	2	3		
1	1	1		

must use

34

$$\binom{8}{4} - \frac{28 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 70$$

$$- \binom{4}{2} \binom{4}{2} - 6 \cdot 6$$

1	1	2	4	7
1		1	2	3
1	1	1	1	1
1	1	1	1	1

avoid

1	3	6	10	15
1	2	3	4	5
1	1	1	1	1
1	1	1	1	1

all

1	2	4	6	8
1	2	2	2	2
1	1	1	1	1
1	1	1	1	1

use

$$\binom{6}{2} - \binom{2}{1} \binom{4}{1}$$

$$\frac{6 \cdot 5}{2 \cdot 1} - 2 \cdot 4$$

$$7 = 15 - 8$$

1	2	4	4	7
1	1	2	B	3
1	A	1	2	3
1	1	1	1	1

\emptyset = no properties or more

A = at least property A

B = at least property B

AB = at least properties A and B

$$\text{avoid} = \emptyset - A - B + AB$$

$$7 = 35 - 20 - 20 + 12$$

1	4	10	20	35
1	3	6	10	15
1	2	3	4	5
1	1	1	1	1

\emptyset

2	6	12	20
2	4	6	8
1	2	2	2
1	1	1	1

A

1	10	20
1	3	6
1	2	3
1	1	1

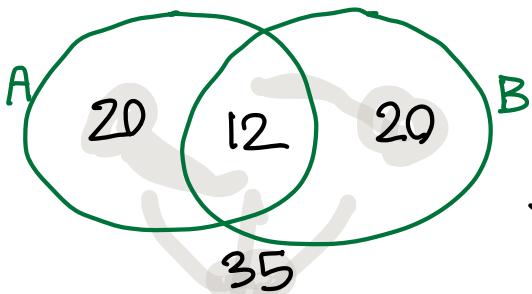
B

1	6	12
1	2	4
1	2	2
1	1	1

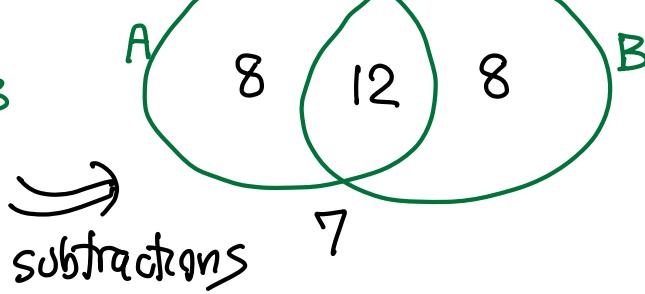
AB

Venn diagram

at least



exactly



subtractions

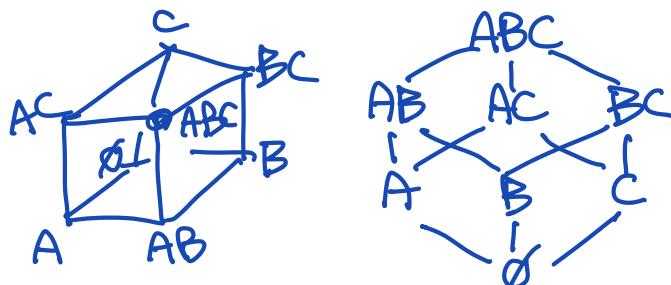
$$\begin{matrix} \emptyset \\ A \\ B \\ AB \end{matrix} \left[\begin{matrix} 35 \\ 20 \\ 20 \\ 12 \end{matrix} \right] = \left[\begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{matrix} \right] \left[\begin{matrix} 7 \\ 8 \\ 8 \\ 12 \end{matrix} \right]$$

we know at least

we want exactly

Inclusion-exclusion
is scaling matrix

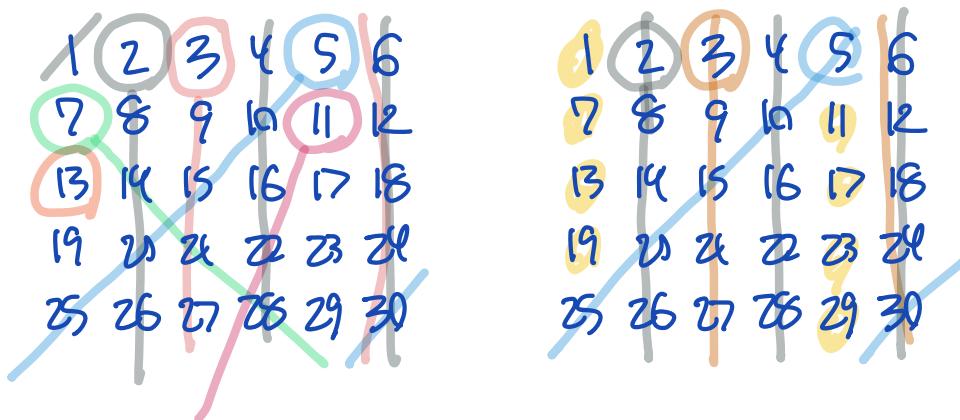
Inclusion-exclusion
Möbius inversion



$$\begin{matrix} x^2 & xy & y^2 \\ A & B & C \end{matrix}$$

$$AC = ABC$$

How many integers 1..30 not divisible by 2, 3, or 5



properties $A = \text{divisible by } 2$
 $B = \text{divisible by } 3$
 $C = \text{divisible by } 5$

$$\emptyset - A - B - C + AB + AC + BC - ABC$$

\emptyset	+								1
A	+	-							0
AB	+	-	-			+			0
ABC	+	-	-	-	-	+	+	-	0

1..30

$$\emptyset - A - B - C + AB + AC + BC - ABC$$

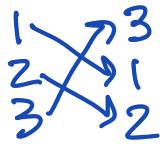
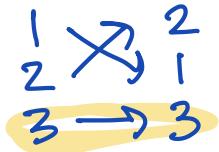
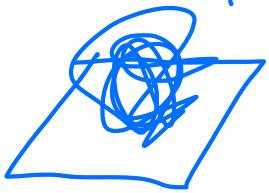
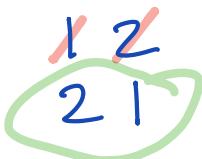
$$\frac{30}{1} \quad \frac{30}{2} \quad \frac{30}{3} \quad \frac{30}{5} \quad \frac{30}{6} \quad \frac{30}{10} \quad \frac{30}{15} \quad \frac{30}{30}$$

$$8 = \frac{30}{30} - \underbrace{\frac{-15}{-15} - \frac{-10}{-10} - \frac{-6}{-6}}_{-31} + \frac{+5}{+5} + \frac{+3}{+3} + \frac{+2}{+2} - \frac{-1}{-1}$$

all multiples of

Hat check problem

How many permutations have no fixed points?

 $n=2$  $n=3$

1	2	3
1	3	2
2	1	3
2	3	1
3	1	2
3	2	1

 $n=4$

1	2	3	4
1	2	4	3
1	3	2	4
1	3	4	2
1	4	2	3
1	4	3	2
2	1	3	4
2	1	4	3
2	3	1	4
2	3	4	1
2	4	1	3
2	4	3	1
3	1	2	4
3	1	4	2
3	2	1	4
3	2	4	1
3	4	1	2
3	4	2	1
4	1	2	3
4	1	3	2
4	2	1	3
4	2	3	1
4	3	1	2
4	3	2	1

Find formula for n agrees with $\frac{n}{\#}$

n	2	3	4
$\#$	1	2	9

$\frac{1}{2} \frac{1}{3} \frac{9}{24}$