Combinatorics #4
January 27, 2022

Derangements (hat-check problem)

1 2 3 4	2 1 3 4	3 1 2 4	4 1 2 3
1 2 4 3	2 1 4 3	3 1 4 2	4 1 3 2
1 3 2 4	2 3 1 4	3 2 1 4	4 2 1 3
1 3 4 2	2 3 4 1	3 2 4 1	4 2 3 1
1 4 2 3	2 4 1 3	3 4 1 2	4 3 1 2
1 4 3 2	2 4 3 1	3 4 2 1	4 3 2 1

$n=4$ $\boxed{9}$

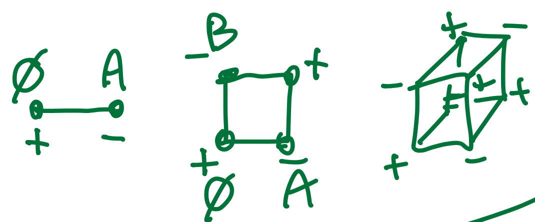
Inclusion-exclusion

Properties
A 1 in position 1
B 2 in position 2
C 3 in position 3
D 4 in position 4

$\geq\emptyset$ at least no props
$\geq A$ at least prop A
$\geq AB$ " " A and B
$\#A$ = exactly A no more
$\#\emptyset$

$$\#\emptyset = \geq\emptyset - \geq A - \geq B - \geq C - \geq D + \geq AB + \dots$$

$$\#\emptyset = \geq\emptyset - \geq\{A,B,C,D\} + \geq\{AB, AC, AD, BC, BD, CD\} - \geq\{ABC, ABD, ACD, BCD\} + \geq ABCD$$



any n

1 property

$\boxed{1\ |\ |\ |\ }$ $\boxed{\ |2|3|\ }$

n properties

$\binom{4}{5} = \frac{\cancel{4\cdot3\cdot2\cdot1\cdot0}}{5!}$

$$\emptyset - \geq\{A,B,C,D\} + \geq\{AB,AC,AD,BC,BD,CD\}$$

$$n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \binom{n}{4}(n-4)! -$$

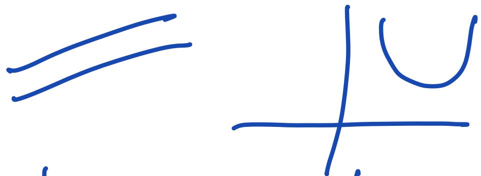
$$= n! - \frac{n}{1}(n-1)! + \frac{n(n-1)}{2\cdot1}(n-2)! - \frac{n(n-1)(n-2)}{3!}(n-3)! + \frac{(\dots)}{4!}(n-4)! -$$

$$= n!\left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - (\cdot)\right) = n!\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}$$

$$\approx \left[\frac{n!}{e}\right]$$

$$e^x = \sum_{j=0}^{\infty}\frac{x^j}{j!} \qquad e^{-1} = \sum_{j=0}^{\infty}\frac{(-1)^j}{j!} \qquad \frac{d}{dx}e^x = e^x$$

Algebraic Geometry



$x^2+1=0$

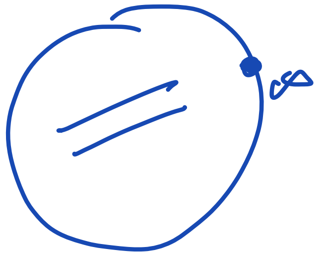
$\mathbb{C}$ $i, -i$

studies zerosets of polynomial equations

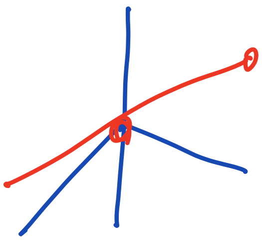
projective space of ratios

$x:y$ $x:1$ x ∞ $1:0$

$\mathbb{P}^2$ $x:y:z$



$2:3 = 4:6$



$(x,y,z) \sim (\lambda x, \lambda y, \lambda z)$ for any λ

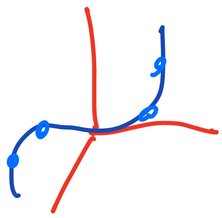
homogeneous polynomials $x^2+xy+y^2 \to \lambda^2(x^2+xy+y^2)$

$x=\lambda x$ $y=\lambda y$ $z=\lambda z$

$\mathbb{R}^1 \longrightarrow \mathbb{R}^3$

$\gamma(t) = (t, t^2, t^3)$

twisted cubic curve



$b^2-ac=0$

$(s^2t)^2-(s^3)(st^2)=0$

$$\gamma(s,t) = (\overset{a}{s^3}, \overset{b}{s^2t}, \overset{c}{st^2}, \overset{d}{t^3})$$

$\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$

$$\left\{\begin{array}{l} b^2-ac=0 \\ bc-ad=0 \\ c^2-bd=0 \end{array}\right\}$$

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\mathbb{R}^4$ $4-2=2$

$3d+1$

1	$\{a,b,c,d\}$	$\{a^2,b^2,c^2,d^2,ab,ac,ad,bc,bd,cd\}$	
$\mathbb{R}^1$ $d=0$	$\mathbb{R}^4$ $d=1$	$\mathbb{R}^{10}$ $d=2$	$\mathbb{R}^{20}$ $d=3$
1	4	7	10

$\mathbb{P}^0$ • ratios in x $\qquad 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8} \to 2 = \frac{1}{1-\frac{1}{2}}$

$1 \quad x \quad x^2 \quad x^3 \quad x^4 \quad x^5 + \cdots$ $\qquad$ generating functions

$f(d) = \#$ terms deg d in our variables $\qquad f(n) = \ldots$

$\mathbb{P}^0 \quad n=1 \quad x \qquad f(d)=1 \qquad \sum_{d=0}^{\infty} f(d)\,t^d = \sum_{d=0}^{\infty} t^d \qquad \sum_{n=0} F(n)x^n$

$$1+t+t^2+\cdots = \boxed{\frac{1}{1-t}}$$

$\mathbb{P}^1$ ratios in x,y

1	x,y	x^2,xy,y^2	x^3,x^2y,xy^2,y^3
1	2	3	4

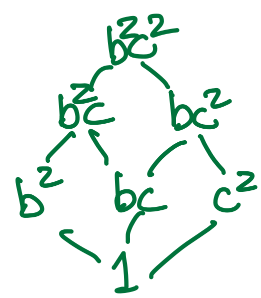
$f(d) = d+1 \qquad \sum_{d=0}^{\infty} f(d)\,t^d = \frac{1}{(1-t)^2}$

$$(1+x+x^2+x^3+\cdots)(1+y+y^2+y^3+\cdots)$$
$$= 1+x+y+x^2+xy+y^2+\cdots \qquad \Big\downarrow \begin{array}{l} x=t \\ y=t \end{array}$$
$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-y}\right) \longrightarrow \frac{1}{(1-t)^2}$$

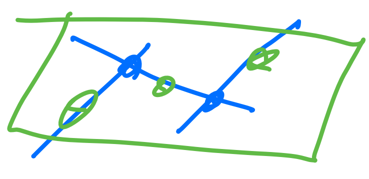
How many mononomials of degree D in a,b,c,d are not multiples of b^2 or bc or c^2?

$$\frac{1-3t^2+2t^2}{(1-t)^4} = \frac{3}{(1-t)^2} - \frac{2}{(1-t)}$$

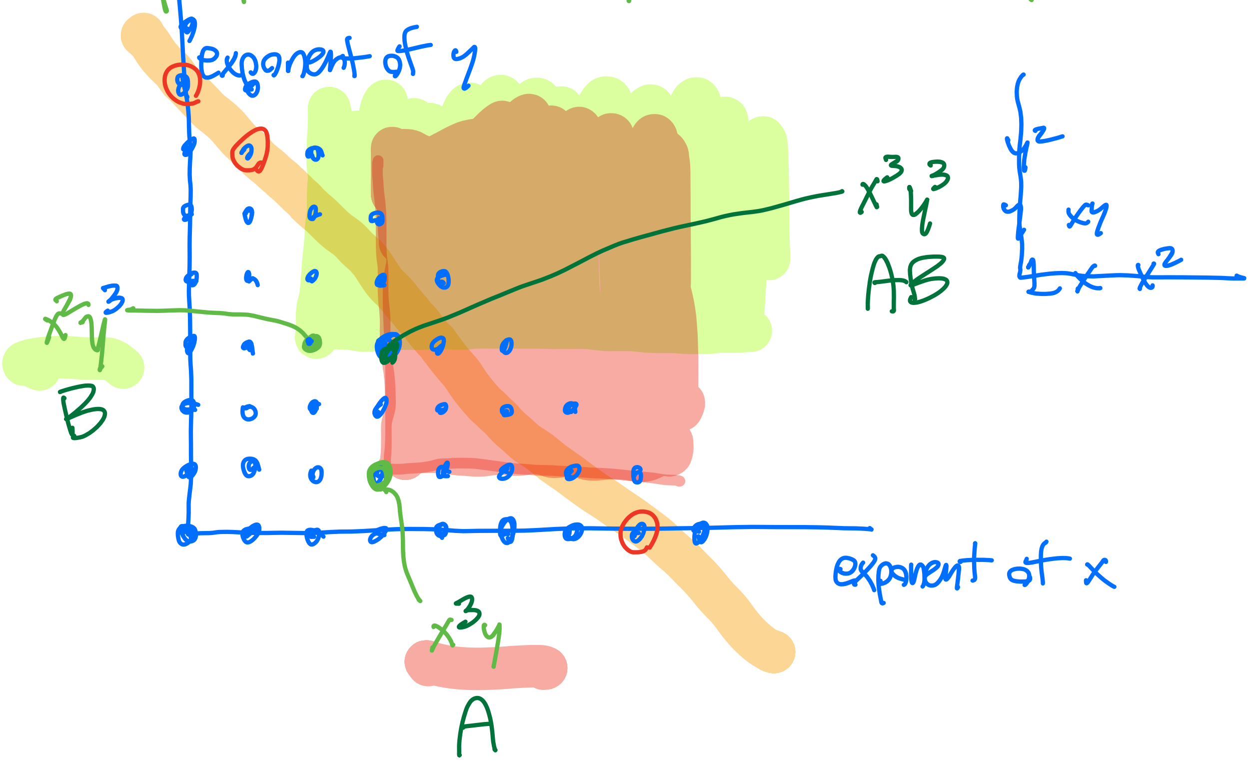
$3\mathbb{P}^1 - 2\mathbb{P}^0$



$$\left\{ \begin{array}{l} b^2-ac=0 \\ bc-ad=0 \\ c^2-bd=0 \end{array} \right\}$$ how much stuff?



~~polynomials~~ terms in x, y not multiples of x^3y or x^2y^3



$$\frac{1}{(1-t)^2} - \frac{t^4}{(1-t)^2} - \frac{t^5}{(1-t)^2} + \frac{t^6}{(1-t)^2} = \frac{3}{(1-t)}$$

$\Sigma\varnothing$ $\qquad \underbrace{(1+x+x^2+\dots)}_{\frac{1}{1-x}}\underbrace{(1+y+y^2+\dots)}_{\frac{1}{1-y}} \xRightarrow[y=t]{x=t} \frac{1}{(1-t)^2}$

ΣA $\qquad x^3y\underbrace{(1+x+x^2+\dots)}_{\frac{1}{1-x}}\underbrace{(1+y+y^2+\dots)}_{\frac{1}{1-y}} \Rightarrow \frac{t^4}{(1-t)^2}$

all multiples of x^3y