Exam 1

Combinatorics, Dave Bayer, February 16-20, 2022

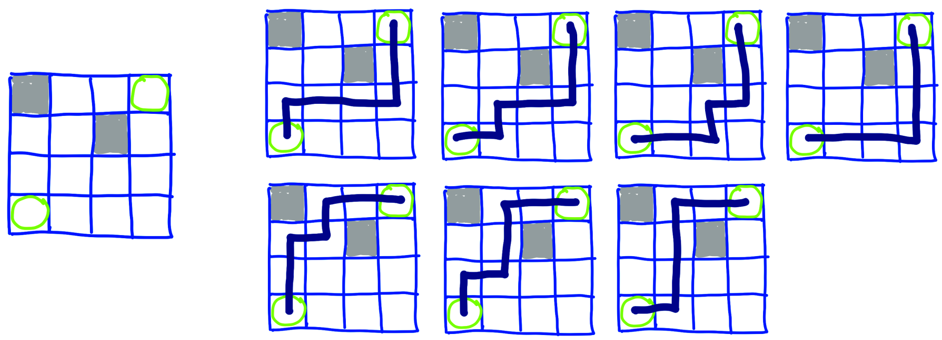
Please show all of your work. You will be graded for both your answers and your explanations. You need not complete the entire exam; some questions are intended to be challenging.

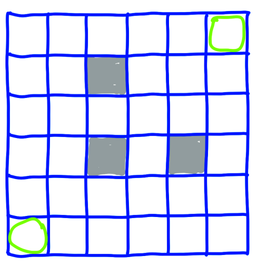
This test is open-book. You may use any resource such as my course materials, textbooks, or *The On-Line Encyclopedia of Integer Sequences*. You may not receive help from another person.

"What can you say about $f(n)$*?"* is up to you. There might be a formula. There might be a generating function. You might notice a pattern, or recognize the sequence.

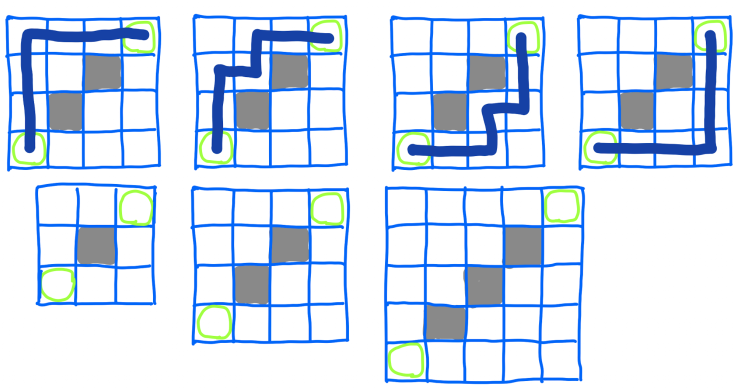
Please match your understanding of my words with the examples, and contact me if you're concerned about any ambiguity.

[1] Shown are two grids with shaded obstacles. We are counting paths that start in the lower left square, end in the upper right square, and step either up or to the right, avoiding the obstacles. For the smaller grid there are 7 paths. How many paths are there, for the larger gird?

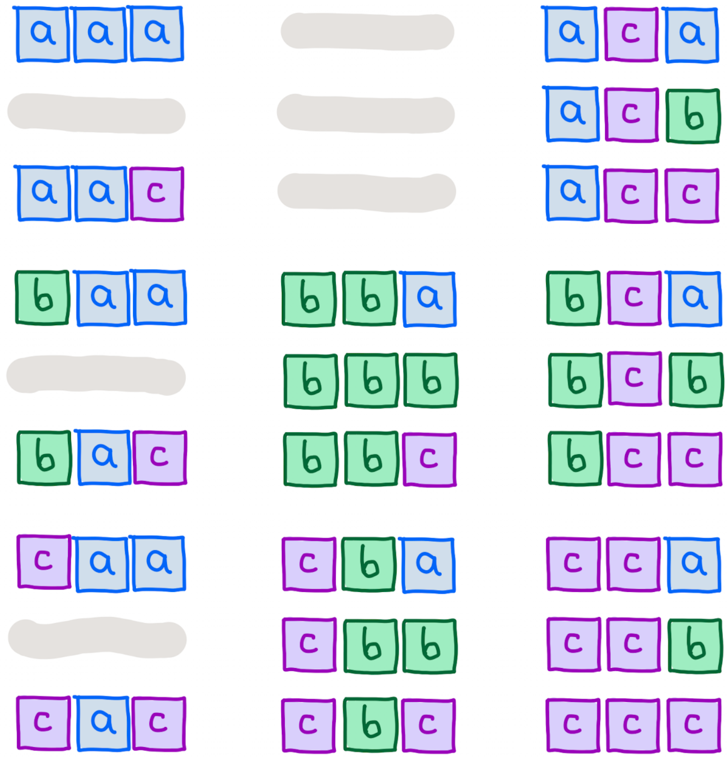




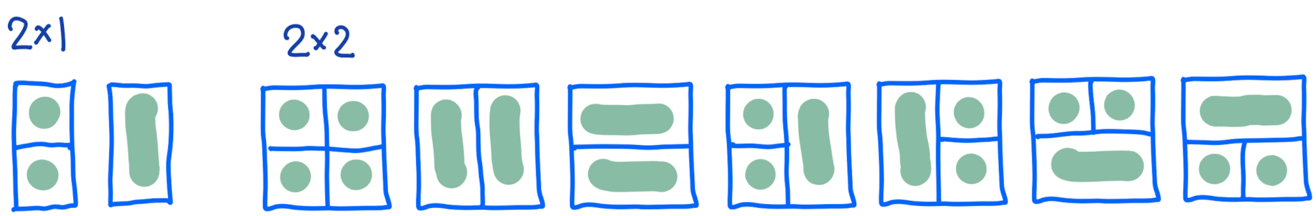
[2] Count paths as before. Let $f(n)$ be the number of paths that avoid the diagonal squares on an $n \times n$ grid, except at the start and the end. As shown below, $f(4) = 4$. Find $f(5)$ and $f(6)$. What can you say about $f(n)$?



[3] Let $f(n)$ be the number of words of length n in the alphabet $\{a, b, c\}$ with the property that b never immediately follows a. As shown below, $f(3) = 21$. Find $f(4)$ and $f(5)$. What can you say about $f(n)$?



[4] Let $f(n)$ be the number of ways of arranging 1×1 tiles and 1×2 tiles in a $2 \times n$ grid. As shown below, $f(1) = 2$ and $f(2) = 7$. Find $f(3)$ and $f(4)$. What can you say about $f(n)$?



[5] Let $f(n)$ be the number of ways of placing three markers on an $n \times n$ board so no two markers are side by side, either vertically or horizontally. As shown below, $f(3) = 22$. Find $f(4)$. What can you say about $f(n)$?