Exam 1

Combinatorics, Dave Bayer, February 16-20, 2022

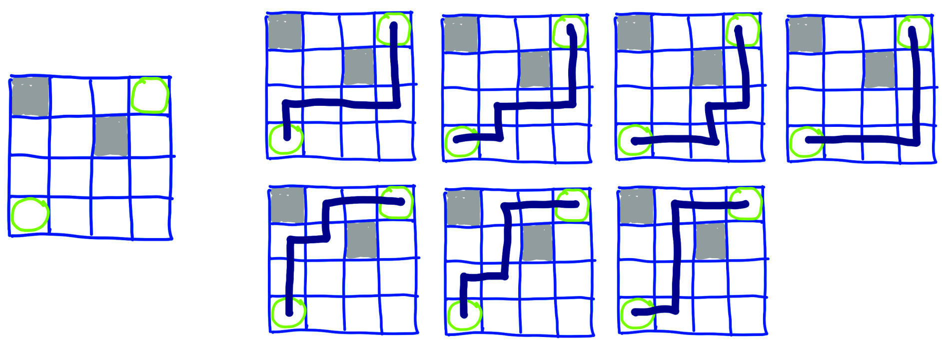
Please show all of your work. You will be graded for both your answers and your explanations. You need not complete the entire exam; some questions are intended to be challenging.

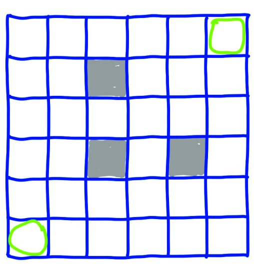
This test is open-book. You may use any resource such as my course materials, textbooks, or *The On-Line Encyclopedia of Integer Sequences*. You may not receive help from another person.

"What can you say about $f(n)$*?"* is up to you. There might be a formula. There might be a generating function. You might notice a pattern, or recognize the sequence.

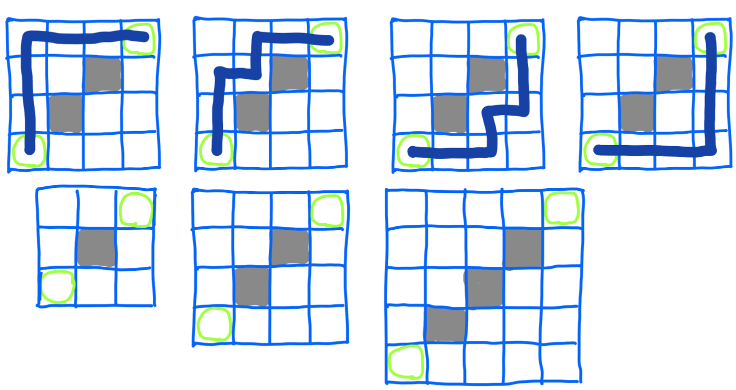
Please match your understanding of my words with the examples, and contact me if you're concerned about any ambiguity.

[1] Shown are two grids with shaded obstacles. We are counting paths that start in the lower left square, end in the upper right square, and step either up or to the right, avoiding the obstacles. For the smaller grid there are 7 paths. How many paths are there, for the larger gird?





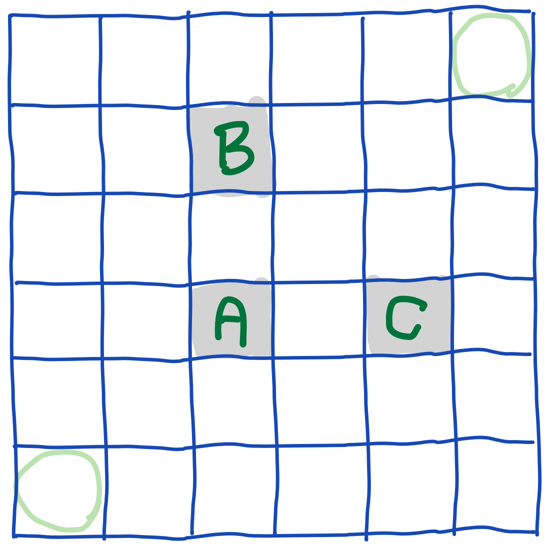
[2] Count paths as before. Let $f(n)$ be the number of paths that avoid the diagonal squares on an $n \times n$ grid, except at the start and the end. As shown below, $f(4) = 4$. Find $f(5)$ and $f(6)$. What can you say about $f(n)$?



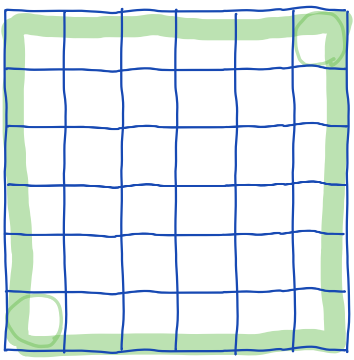
[1]

$\boxed{\text{60 paths}}$

1	6	6	14	30	60
1	5		8	16	30
1	4	4	8	8	14
1	3		4		6
1	2	3	4	5	6
1	1	1	1	1	1

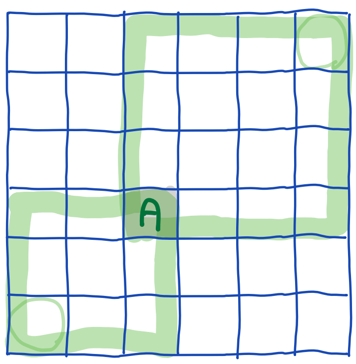


Or by inclusion-exclusion counting:



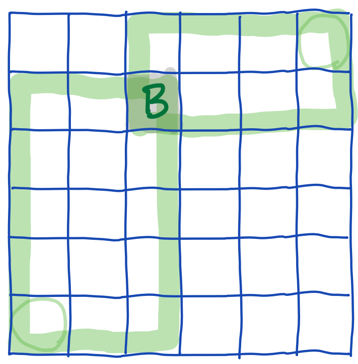
$\binom{10}{5} = \frac{{}^{2}\cancel{10}\cdot 9\cdot\cancel{8}\cdot 7\cdot\cancel{6}\,{}^{2}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}$

$4\cdot9\cdot7 = 252$



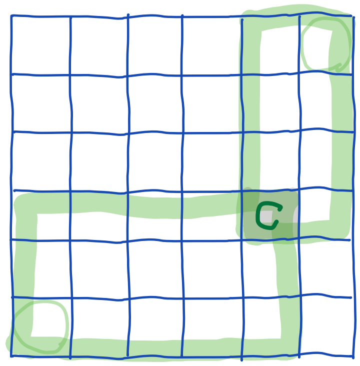
$\binom{4}{2}\binom{6}{3} = 6\cdot20$

-120



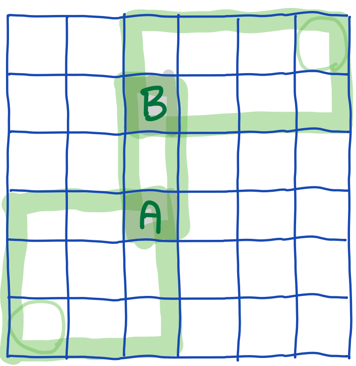
$\binom{6}{2}\binom{4}{3} = 15\cdot4$

-60



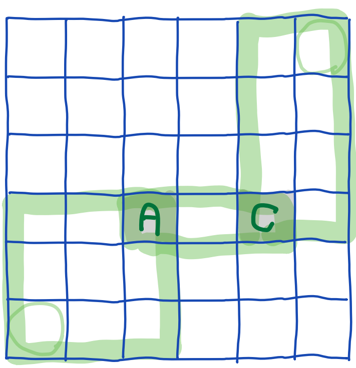
by symmetry

-60



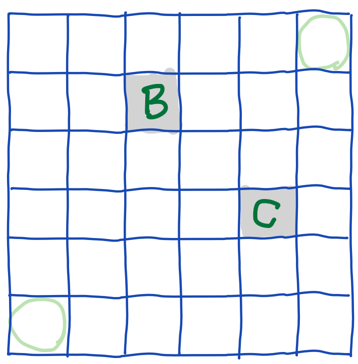
$\binom{4}{2}\binom{2}{0}\binom{4}{3} = 6\cdot1\cdot4$

$+24$



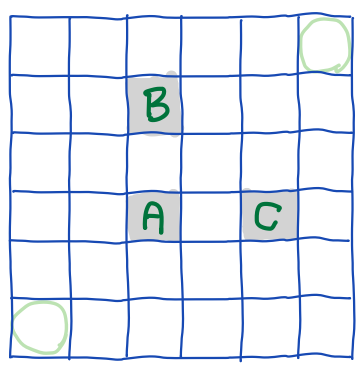
by symmetry

$+24$



0

not possible

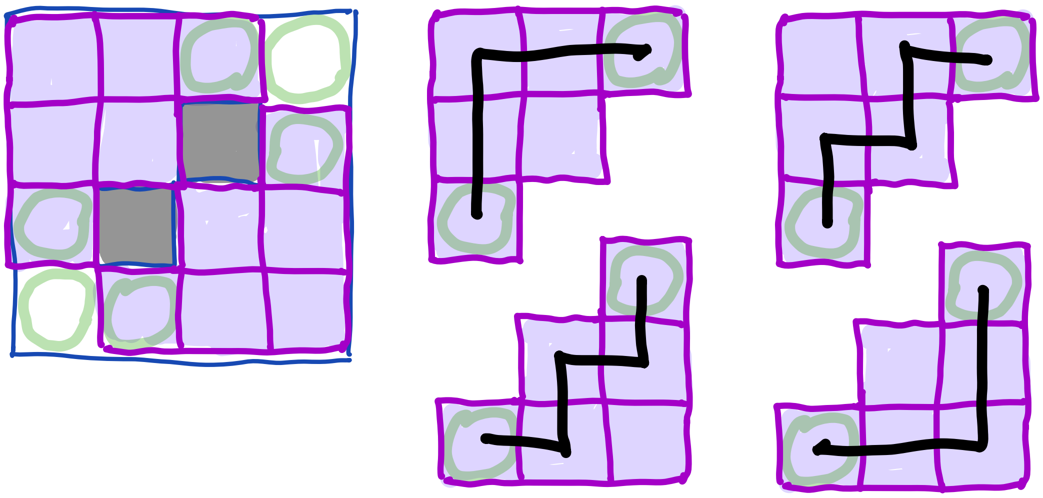


0

not possible

$252 - 120 - 60 - 60 + 24 + 24 = 60$ ✓

[2] $F(4)=4$



The paths on either side of diagonal are Catalan paths. So twice Catalan numbers.

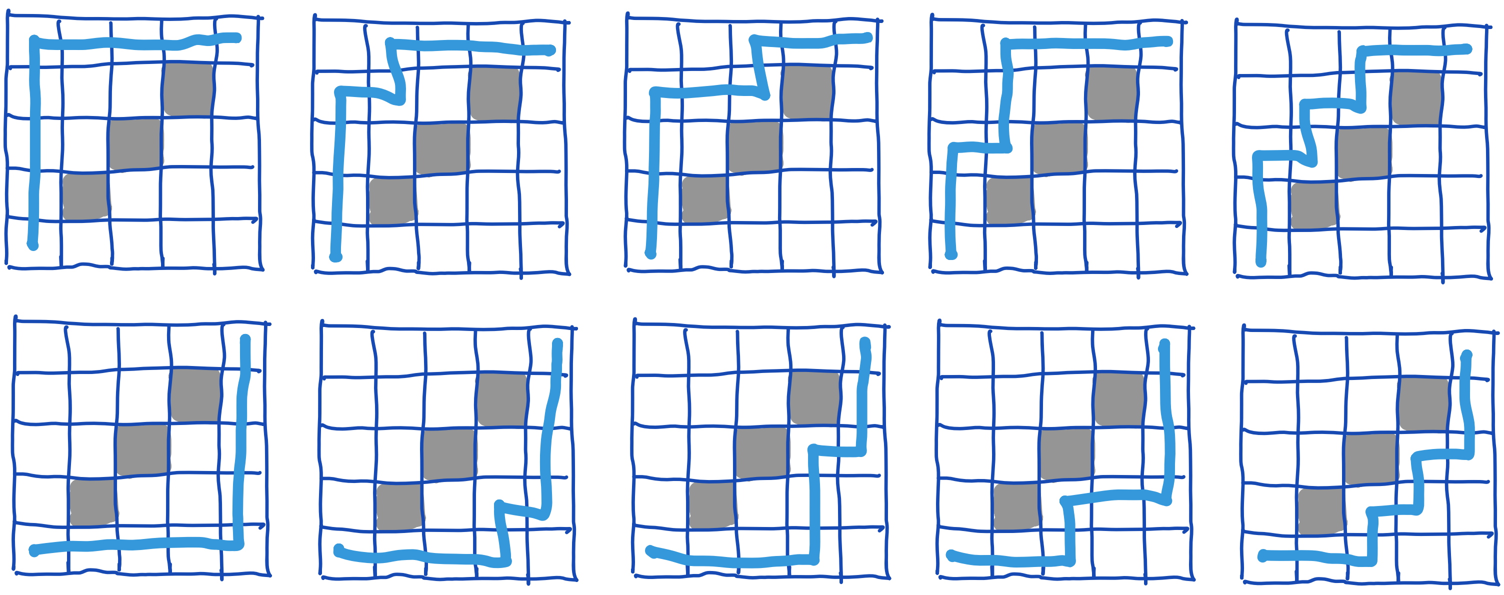
$$\boxed{\begin{aligned}F(5)&=10\\F(6)&=28\end{aligned}}$$

$$(1,1,2)\cdot(2,1,1)=5$$

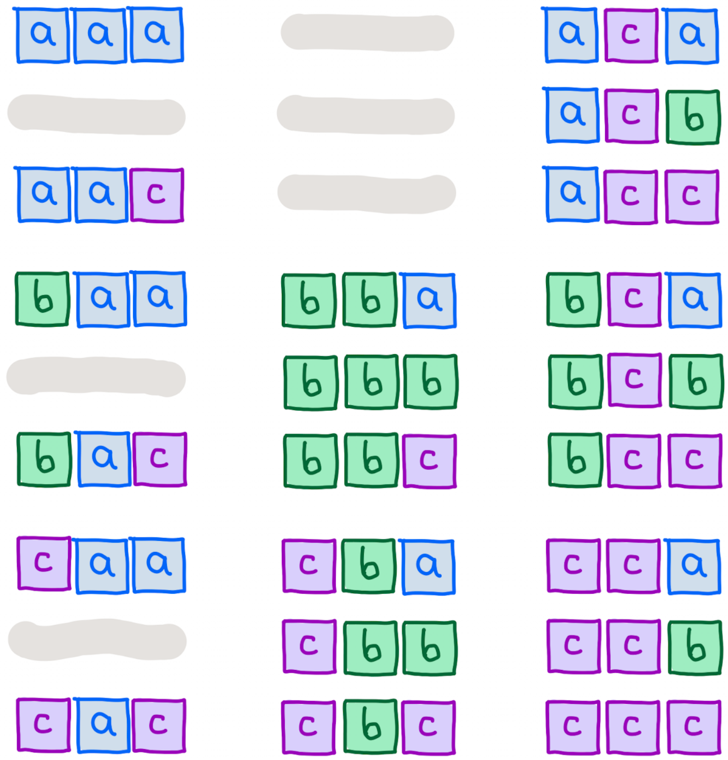
$$(1,1,2,5)\cdot(5,2,1,1)=14$$

OEIS A284016

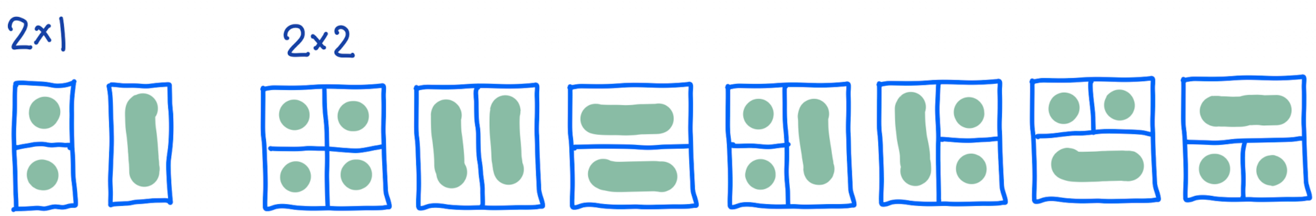
Check $F(5)=10$:



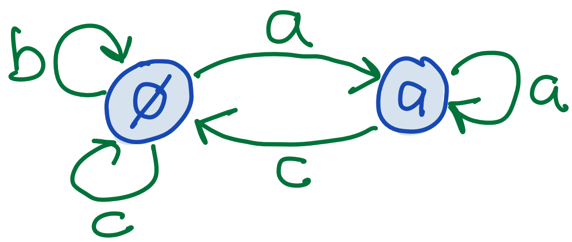
[3] Let $f(n)$ be the number of words of length n in the alphabet $\{a, b, c\}$ with the property that b never immediately follows a. As shown below, $f(3) = 21$. Find $f(4)$ and $f(5)$. What can you say about $f(n)$?



[4] Let $f(n)$ be the number of ways of arranging 1×1 tiles and 1×2 tiles in a $2 \times n$ grid. As shown below, $f(1) = 2$ and $f(2) = 7$. Find $f(3)$ and $f(4)$. What can you say about $f(n)$?



[3] Model as finite state machine



We want to count the total # paths of length n starting at ∅ and ending at ∅ or a.

$$\text{to}\ \begin{array}{c|cc} & ∅ & a \\ \hline ∅ & 2 & 1 \\ a & 1 & 1 \end{array}\ \text{from}$$

n=1

$$f(n) = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{\text{ending at } ∅ \text{ or } a} \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^n}_{n \text{ steps}} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\text{starting at } ∅}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \qquad f(2) = 3+5 = 8$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix} \qquad f(3) = 8+13 = 21$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} 34 & 21 \\ 21 & 13 \end{bmatrix} \qquad f(4) = 21+34 = 55$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 34 & 21 \\ 21 & 13 \end{bmatrix} = \begin{bmatrix} 89 & 55 \\ 55 & 34 \end{bmatrix} \qquad f(5) = 55+89 = 144$$

$\boxed{\text{OEIS A001906}}$

Fibonacci $1 \quad \underbrace{1 \quad 2}_{F(1)} \quad \underbrace{3 \quad 5}_{F(2)} \quad \underbrace{8 \quad 13}_{F(3)} \quad \underbrace{21 \quad 34}_{F(4)} \quad \underbrace{55 \quad 89}_{F(5)}$

(Nice puzzle: Find a bijection between these words and an example of Fibonacci numbers.)

Find recurrence via generating function:

$$F(t) = \sum_{n=0}^{\infty} F(n) t^n = \sum_{n=0}^{\infty} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^n t^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\sum_{n=0}^{\infty} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^n t^n = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} t \right)^{-1}$$

$$= \begin{bmatrix} 1-2t & -t \\ -t & 1-t \end{bmatrix}^{-1} = \begin{bmatrix} 1-t & t \\ t & 1-2t \end{bmatrix} \Big/ (1-3t+t^2)$$

$$(1-2t)(1-t) - t^2 = 1 - 3t + t^2$$

So $\boxed{F(t) = 1/(1-3t+t^2)}$

We read this as $F(n) = \begin{cases} 0, & n<0 \\ 1, & n=0 \\ 3F(n-1) - F(n-2), & n>0 \end{cases}$

Why? $(1-3t+t^2)\,F(t) = 1$

*	1	3t	8t²	21t³	55t⁴	
1	1	$3t$	$8t^2$	$21t^3$	$55t^4$	...
$-3t$	$-3t$	$-9t^2$	$-24t^3$	$-61t^4$	...	
t^2	t^2	$3t^3$	$8t^4$	...		

$\Rightarrow$

*	1	3	8	21	55	
1	1	3	8	21	55	...
-3	-3	-9	-24	-61	...	
1	1	3	8	...		

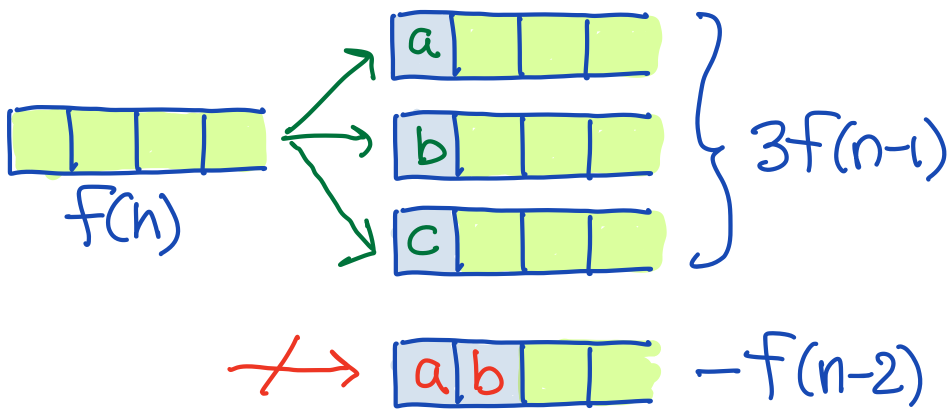
$f(0) = 1$, $\quad f(n) - 3f(n-1) + f(n-2) = 0$

$1 \quad -3t \quad +t^2$

We can understand $f(n) = 3f(n-1) - f(n-2)$

as

allow everything



but subtract off forbidden ab

Another approach: Consider all words in a, b, c and the length 2 ligature $\underline{ab}$

There are 2^k ways to write a word with k copies of ab:

$a \mid b \mid c \mid a \mid b$	$\underline{ab} \mid c \mid a \mid b$	$a \mid b \mid c \mid \underline{ab}$	$\underline{ab} \mid c \mid \underline{ab}$
	$-$	$-$	$-$ $-$
$+$	$-$	$-$	$+$

We can get these all to cancel out by making each $\underline{ab}$ minus, leaving only allowed words.

A generating function that writes out all allowed words, and cancels out all forbidden words, is the geometric series

$$1 + (a+b+c-\underline{ab}) + (a+b+c-\underline{ab})^2 + \dots$$

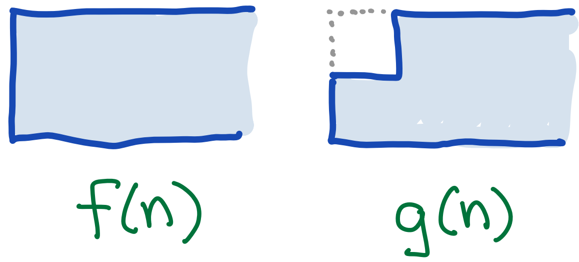
$$1 \quad a \quad b \quad c \quad \cancel{-\underline{ab}} \quad aa \quad \cancel{ab} \quad ac \quad -a\underline{ab} \quad ba \quad bb \dots$$

$$= \sum_{m=0}^{\infty} (a+b+c-\underline{ab})^m = 1/(1-a+b+c+\underline{ab})$$

Setting $a=b=c=t$ and $\underline{ab} = t^2$ we get

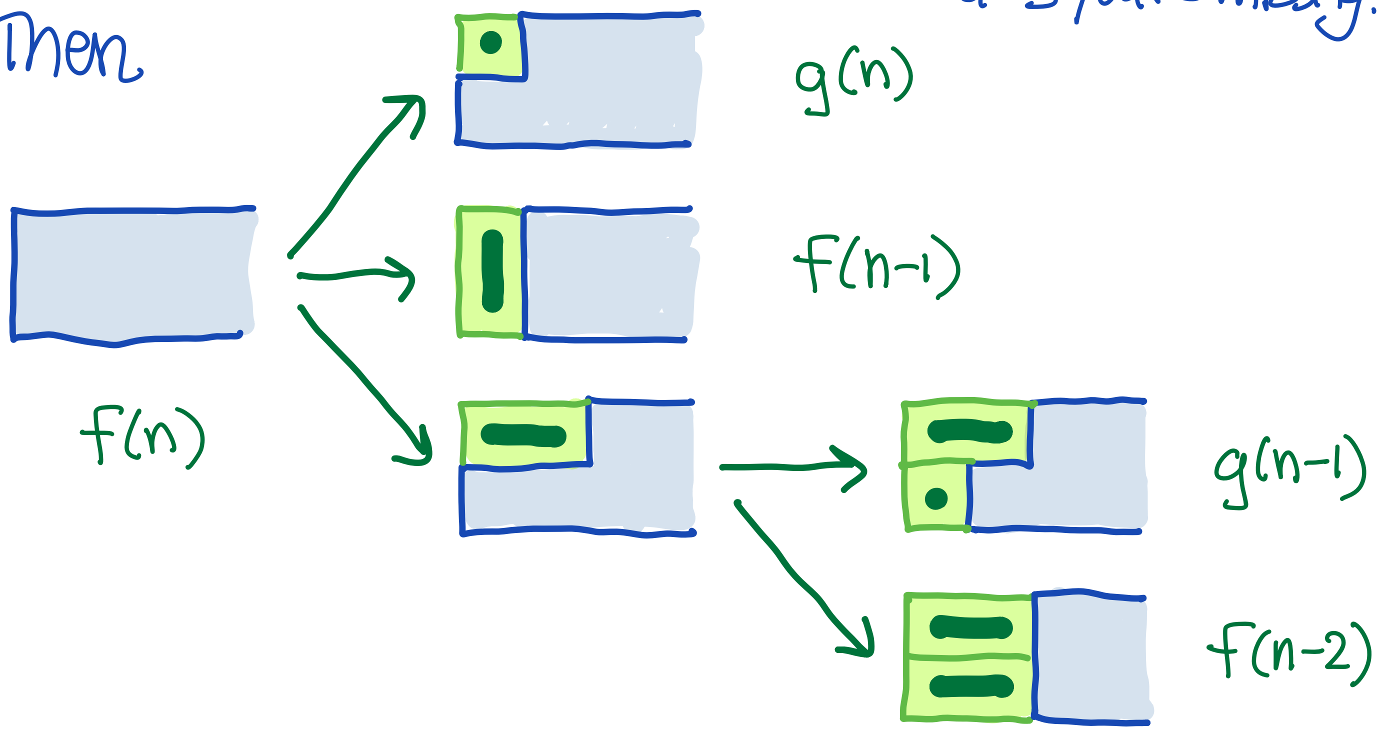
$\boxed{F(t) = 1/(1-3t+t^2)}$ as before.

[4]



Let f(n) count tilings of the 2×n grid, and
let g(n) count tilings of the helper grid with
a square missing.

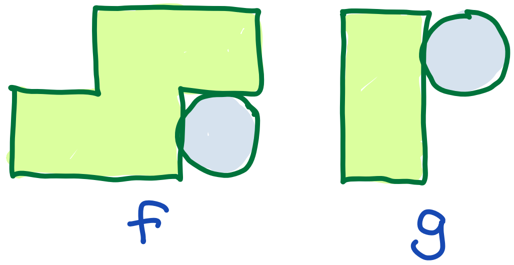
Then

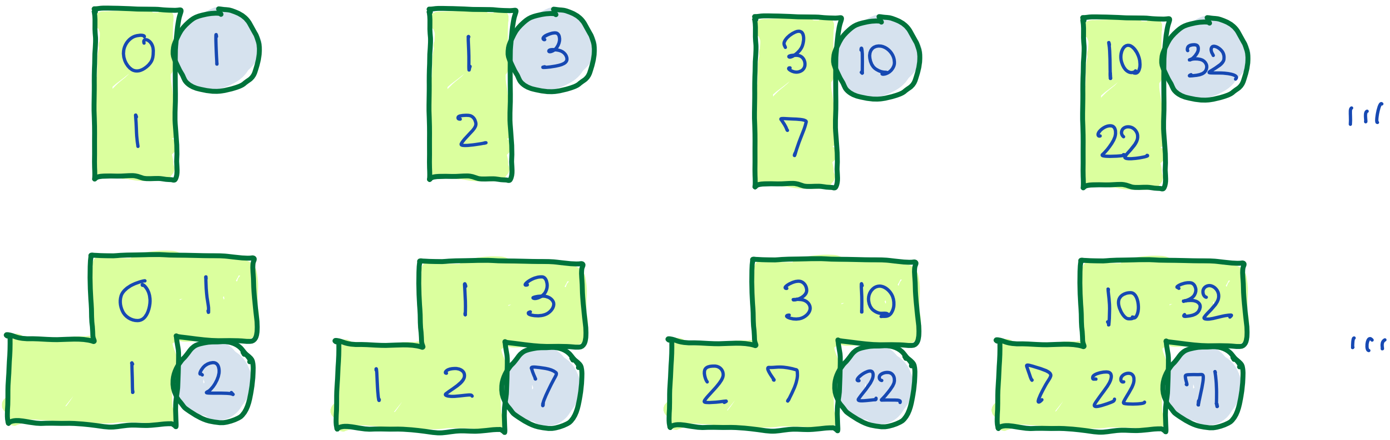


So

$$\boxed{\begin{aligned} f(n) &= f(n-1) + f(n-2) + g(n) + g(n-1) \\ g(n) &= f(n-1) + g(n-1) \end{aligned}}$$

n	0	1	2	3	4
$g(n)$	0	1	3	10	32
$f(n)$	1	2	7	22	71





Now rewrite as generating functions:

$$F(t) = 1 + tF(t) + t^2F(t) + G(t) + tG(t)$$
$$G(t) = tF(t) + tG(t)$$

$$\begin{bmatrix} 1-t-t^2 & -1-t \\ -t & 1-t \end{bmatrix}\begin{bmatrix} F(t) \\ G(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} F(t) \\ G(t) \end{bmatrix} = \begin{bmatrix} 1-t & 1+t \\ t & 1-t-t^2 \end{bmatrix} \Big/ (1-3t-t^2+t^3) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(1-t-t^2)(1-t) - (-t)(-1-t) = 1-3t-t^2+t^3$$

$1-6-4+8 = -1$ ✓

$$\begin{array}{cccc} 1 & t & t^2 & t^3 \\ \hline + & - & - & \\ & - & + & + \\ & - & - & \end{array}$$

$$\left|\begin{bmatrix} -5 & -3 \\ -2 & -1 \end{bmatrix}\right| = -1$$

check $t=2$

So $\boxed{F(t) = \dfrac{1-t}{1-3t-t^2+t^3}}$

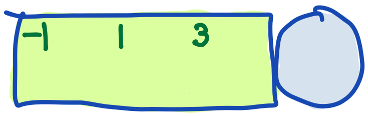
or $(1-3t-t^2+t^3)\,F(t) = 1-t$

$$\begin{array}{r|rrrrr}
 & 1 & 2 & 7 & 22 & 71 \\
\hline
1 & 1 & 2 & 7 & 22 & 71 \\
-3 & -3 & -6 & -21 & -66 & \\
-1 & -1 & -2 & -7 & & \\
1 & 1 & 2 & & & \\
\hline
\end{array}$$

$f(0) = 1$

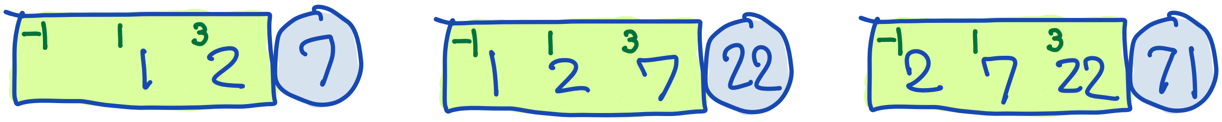
$f(1) = 2$

$f(n) = 3f(n-1) + f(n-2) - f(n-3)$



f

n	0	1	2	3	4
$f(n)$	1	2	7	22	71

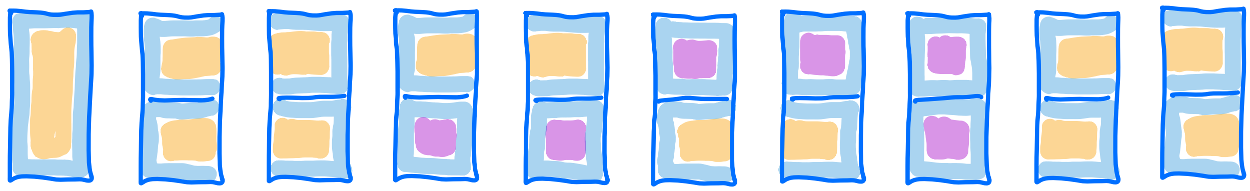


$f(2)$ $\quad$ $f(3)$ $\quad$ $f(4)$

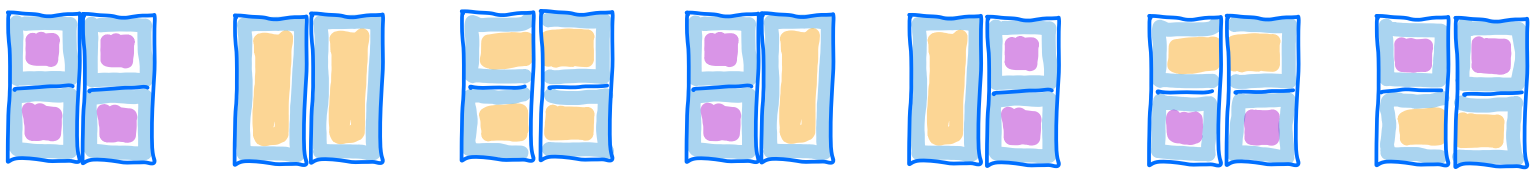
$\boxed{\text{OEIS A030186}}$

Alternate approach:

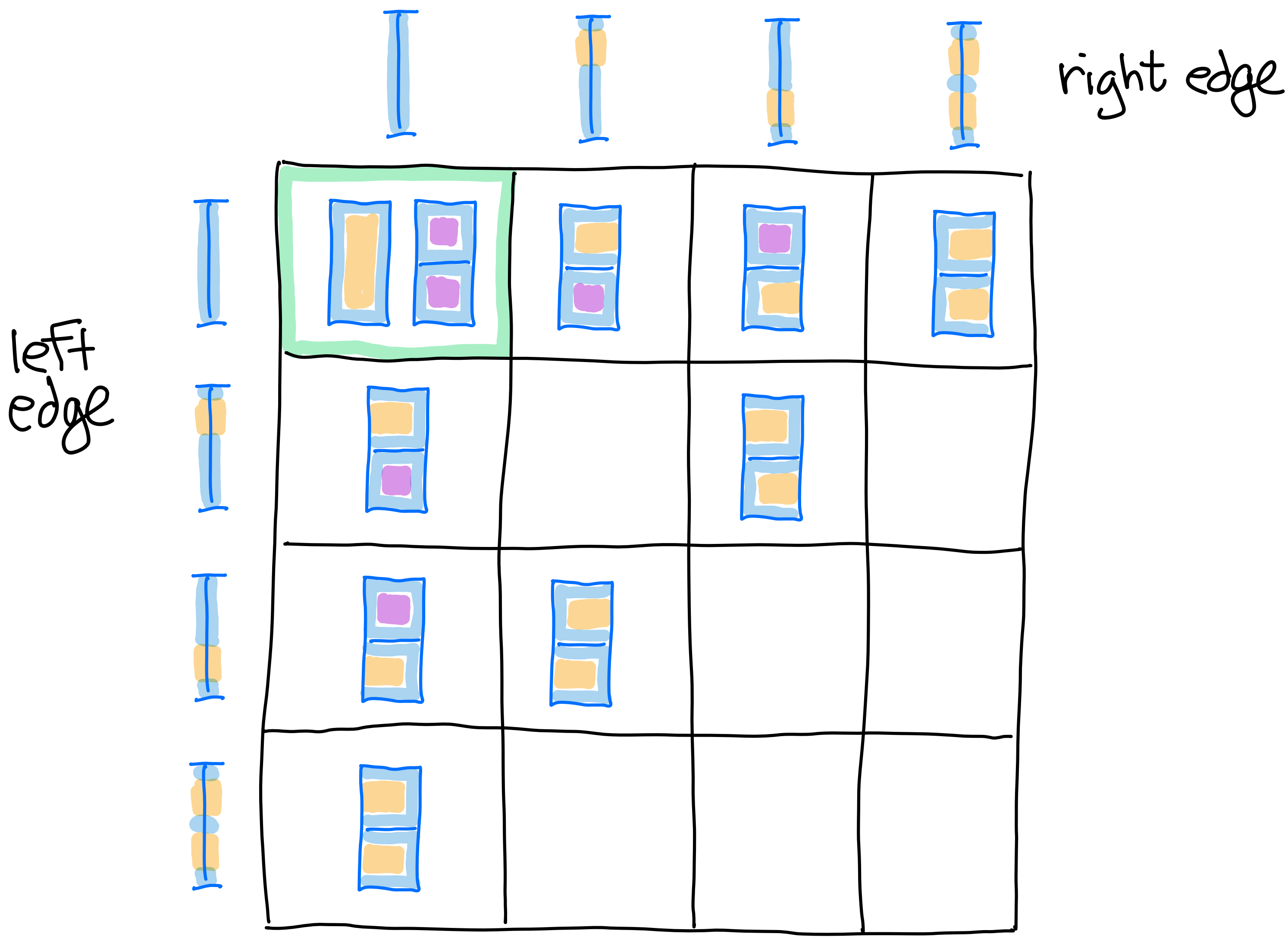
These are length n "words" using the alphabet



The letters have to match up to form valid words:



We get a transition matrix



We can get the diagrams themselves by the $\boxed{(1,1)}$ entry in powers of this matrix of drawings.

For just the counts $F(n)$,

$$\underset{A}{\begin{bmatrix} \boxed{2} & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}} \quad \underset{A^2}{\begin{bmatrix} \boxed{7} & 3 & 3 & 2 \\ 3 & 2 & 1 & 1 \\ 3 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix}} \quad \underset{A^3}{\begin{bmatrix} \boxed{22} & 10 & 10 & 7 \\ 10 & & & \\ 10 & & \text{skip} & \\ 7 & & & \end{bmatrix}} \quad \underset{A^4}{\begin{bmatrix} \boxed{71} & & & \\ & & \text{skip} & \\ & & & \\ & & & \end{bmatrix}}$$

For the generating function $F(t) = \sum_{n=0}^{\infty} F(n) t^n$,

$$\sum_{n=0}^{\infty} A^n t^n = (I - At)^{-1}$$

$$\begin{bmatrix} 1-2t & -t & -t & -t \\ -t & 1 & -t & 0 \\ -t & -t & 1 & 0 \\ -t & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \boxed{F(t)} & & & \\ & & & \\ & & \text{skip} & \\ & & & \end{bmatrix}$$

use formula for inverse

(or use a computer algebra system!)

$$\begin{vmatrix} 1-2t & -t & -t & -t \\ -t & 1 & -t & 0 \\ -t & -t & 1 & 0 \\ -t & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1-2t-t^2 & -t & -t & 0 \\ -t & 1 & -t & 0 \\ -t & -t & 1 & 0 \\ -t & 0 & 0 & 1 \end{vmatrix}$$

Adding t*(row 4) to (row 1) doesn't change determinant

This reduces us to 3×3, keep playing this game:

$$\begin{vmatrix} 1-2t-t^2 & -t & -t \\ -t & 1 & -t \\ -t & -t & 1 \end{vmatrix} = \begin{vmatrix} 1-2t-2t^2 & -t-t^2 & 0 \\ -t-t^2 & 1-t^2 & 0 \\ -t & -t & 1 \end{vmatrix}$$

This reduces us to 2×2

$$\begin{vmatrix} 1-2t-2t^2 & -t-t^2 \\ -t-t^2 & 1-t^2 \end{vmatrix} = (1-2t-2t^2)(1-t^2) - (-t-t^2)(-t-t^2)$$

$$= 1-2t-4t^2+t^4$$

$$\begin{array}{ccccc} 1 & t & t^2 & t^3 & t^4 \\ \hline 1 & -2 & -2 & & \\ & & -1 & 2 & 2 \\ & & -1 & -2 & -1 \\ \hline 1 & -2 & -4 & 0 & 1 \end{array}$$

Now the numerator: $\begin{vmatrix} 1 & -t & 0 \\ -t & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1-t^2$

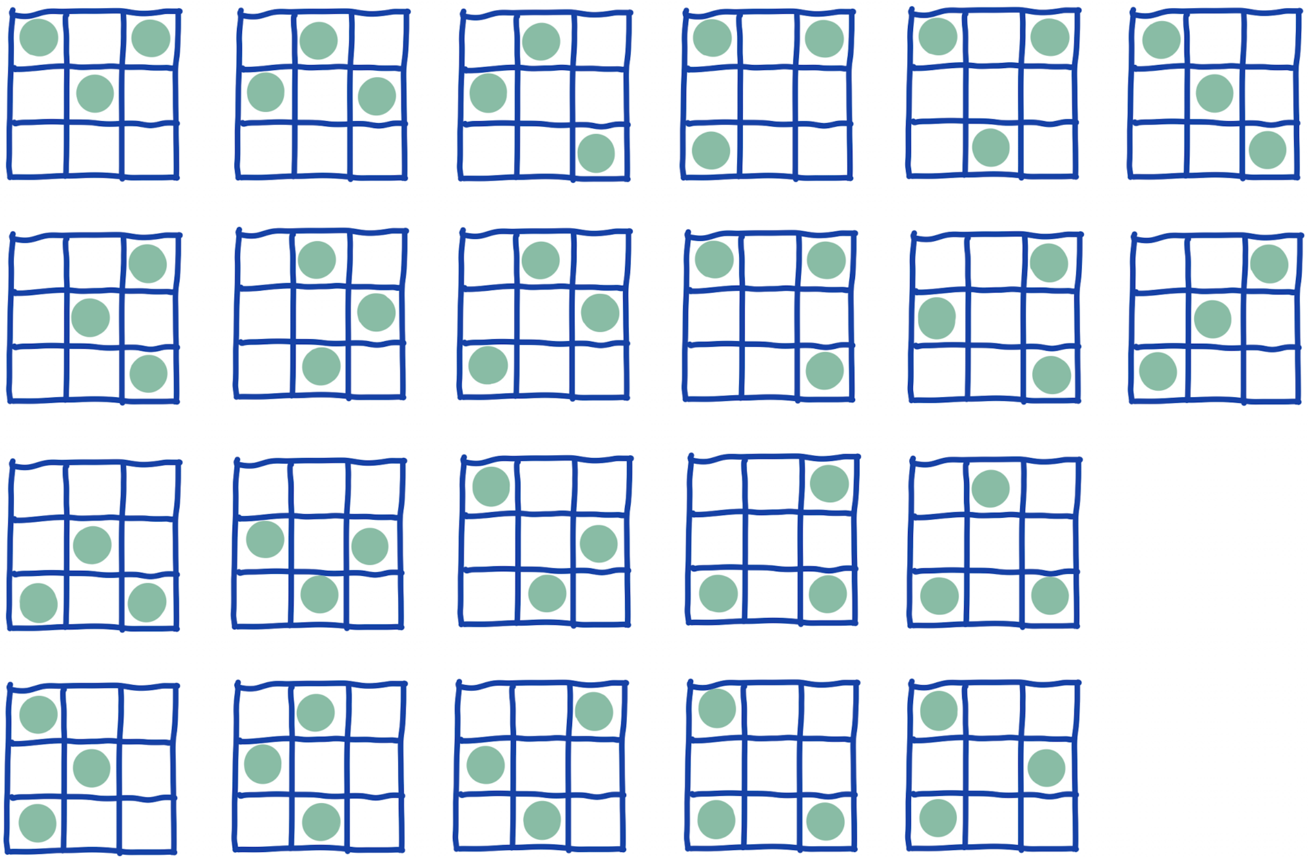
So

$$F(t) = \frac{1-t^2}{1-2t-4t^2+t^4} = \frac{\cancel{(1+t)}(1-t)}{\cancel{(1+t)}(1-3t-t^2+t^3)}$$

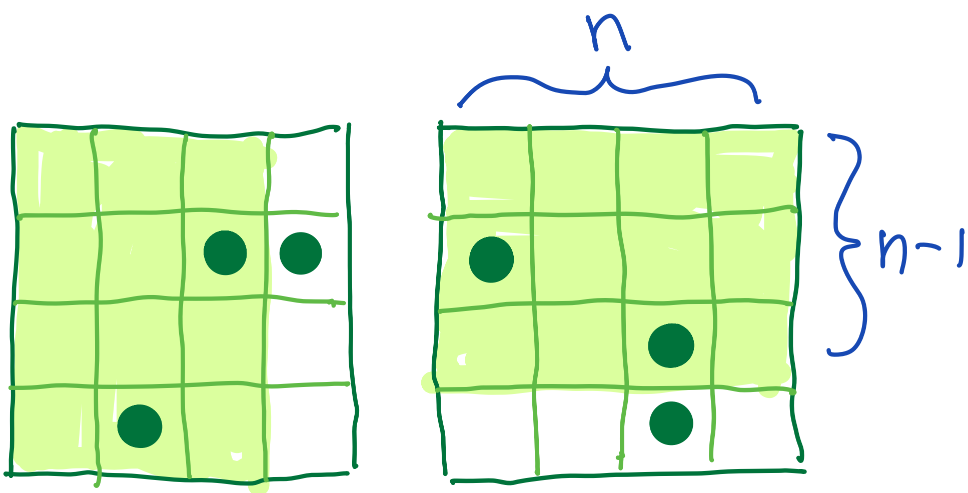
$$\begin{array}{c|cccc|c} & 1 & -3 & -1 & 1 & \\ \hline 1 & 1 & -3 & -1 & 1 & \\ 1 & 1 & -3 & -1 & 1 & \\ \hline 1 & -2 & -4 & 0 & 1 & \end{array} \qquad = \boxed{\frac{1-t}{1-3t-t^2+t^3}}$$

as before

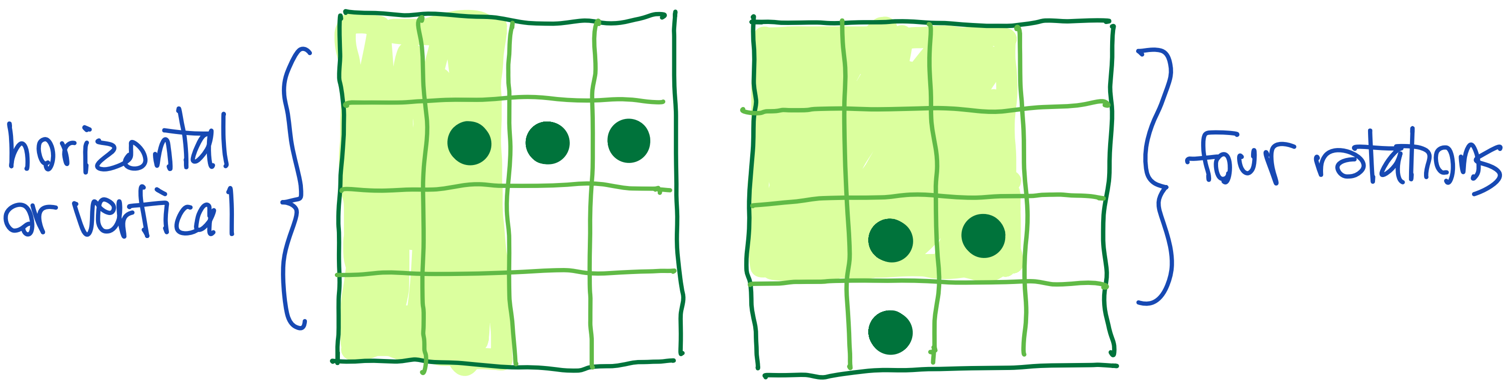
[5] Let $f(n)$ be the number of ways of placing three markers on an $n \times n$ board so no two markers are side by side, either vertically or horizontally. As shown below, $f(3) = 22$. Find $f(4)$. What can you say about $f(n)$?



[5] There are $\binom{n^2}{3}$ ways to place 3 markers on an $n \times n$ board.



There are $2n(n-1)(n^2-2)$ ways to place 3 markers so (at least) one pair is adjacent.



$2n(n-2)$ $\quad$ $4(n-1)^2$

There are two configurations that get subtracted twice, and need to be added back in.

$$f(n) = \binom{n^2}{3} - 2n(n-1)(n^2-2) + 2n(n-2) + 4(n-1)^2$$

$n=2$

$$\binom{4}{3} - 2\cdot 2\cdot 1\cdot 2 + 2\cdot 2\cdot 0 + 4\cdot 1^2$$

$$4 \quad -8 \quad\quad +4 = 0 \checkmark$$

$n=3$

$$\binom{9}{3} - 2\cdot 3\cdot 2\cdot 7 + 2\cdot 3 + 4\cdot 2^2$$

$$\frac{\overset{3}{\cancel{9}}\cdot \overset{4}{\cancel{8}}\cdot 7}{\cancel{3}\cdot\cancel{2}\cdot 1} \qquad 84 - 84 + 6 + 16 = 22 \checkmark$$

$n=4$

$$\binom{16}{3} - 2\cdot 4\cdot 3\cdot 14 + 2\cdot 4\cdot 2 + 4\cdot 3^2$$

$$\frac{\overset{8}{\cancel{16}}\cdot \overset{5}{\cancel{15}}\cdot 14}{\cancel{3}\cdot\cancel{2}\cdot 1} \qquad (8\cdot 5 - 2\cdot 4\cdot 3)14 + 16 + 36$$

$$40 - 24$$

$$16\cdot 14 + 16 + 36$$

$$16\cdot 15 + 36$$

$$276$$

$\boxed{\text{OEIS A172226}}$