Aigner, p242 Burnside's lemma

Counting with symmetry

$$\sum_{x \in X} |G_x| = \sum_{g \in G} |X_g|\,. \qquad (1)$$

Lemma 6.1. *Let G act on X. Then for any $x \in X$,*

$$|M(x)| = \frac{|G|}{|G_x|}\,. \qquad (2)$$

Lemma 6.2 (Burnside–Frobenius). *Let the group G act on X, and let $\mathcal{M}$ be the set of patterns. Then*

$$|\mathcal{M}| = \frac{1}{|G|} \sum_{g \in G} |X_g|\,. \qquad (3)$$

We need to understand how to read this.

X = raw set of objects

G = symmetries acting on X

$\mathcal{M}$ = patterns, equivalence classes of objects up to symmetry

X_g = elements of X fixed by $g \in G$

Example: X = length 2 lists from $\{a,b\}$

$G = \{ 1, \leftrightarrow \}$ (1: do nothing; $\leftrightarrow$: flip)

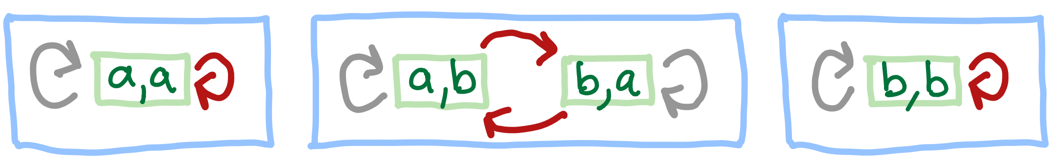
$X = \{ a,a \quad a,b \quad b,a \quad b,b \}$

$\mathcal{M} = \{ [a,a] \quad [a,b \;\; b,a] \quad [b,b] \}$

$|X| = 4$

$|G| = 2$

$|\mathcal{M}| = 3$



$\mathcal{M}$ = "orbits" of action of G on X

$X_1 = \{ a,a \quad a,b \quad b,a \quad b,b \}$

$X_{\leftrightarrow} = \{ a,a \quad b,b \}$

$|X_1| = 4$

$|X_{\leftrightarrow}| = 2$

$$\frac{1}{|G|} \sum_{g \in G} |X_g| = \frac{1}{2}\left(|X_1| + |X_{\leftrightarrow}|\right) = \frac{1}{2}(4+2) = 3 = |\mathcal{M}|$$

Example: X = length 3 lists from $\{a, b, c\}$

$G = \{ 1, \leftrightarrow \}$

1: do nothing

$\leftrightarrow$: flip

$|X| = 27$

$|G| = 2$

$|M| = 18$

$|X_1| = 27$

$|X_{\leftrightarrow}| = 9$

$$\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{2}\left(|X_1| + |X_{\leftrightarrow}|\right) = \frac{1}{2}(27+9) = 18 = |M|$$

Example: X = length k lists from $\{a_1, \dots, a_n\}$

$G = \{ 1, \leftrightarrow \}$

(1 = do nothing, $\leftrightarrow$ = flip)

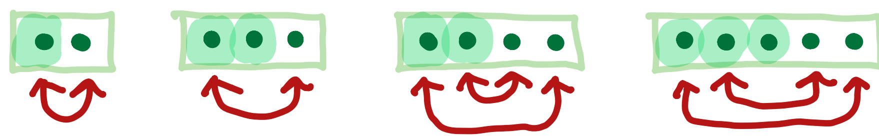
$|X| = n^k = |X_1|$

$|G| = 2$

$|X_{\leftrightarrow}| = n^{\lceil \frac{k}{2} \rceil}$

substep: do a counting problem

$|X_{\leftrightarrow}| = n^{\lceil \frac{k}{2} \rceil}$ ($\lceil \frac{k}{2} \rceil$ = round up $k/2$)



1 choice	2 choices	2 choices	3 choices
n	n^2	n^2	n^3

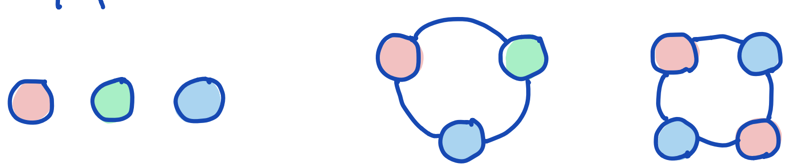
$$\frac{1}{|G|} \sum_{g \in G} |X_g| = \frac{1}{2}(|X_1| + |X_{\leftrightarrow}|) = \frac{1}{2}(n^k + n^{\lceil \frac{k}{2} \rceil}) = |M|$$

$n = k = 2$ $\quad \frac{1}{2}(2^2 + 2) = 3$ ☑

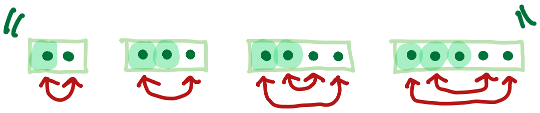
$n = k = 3$ $\quad \frac{1}{2}(3^3 + 3^2) = 18$ ☑

Example: "Necklace" problems

Make an n-bead necklace using k possible colors of beads. Two patterns are the same if they agree after rotation. How many patterns?



For each n, there will be a version of the "

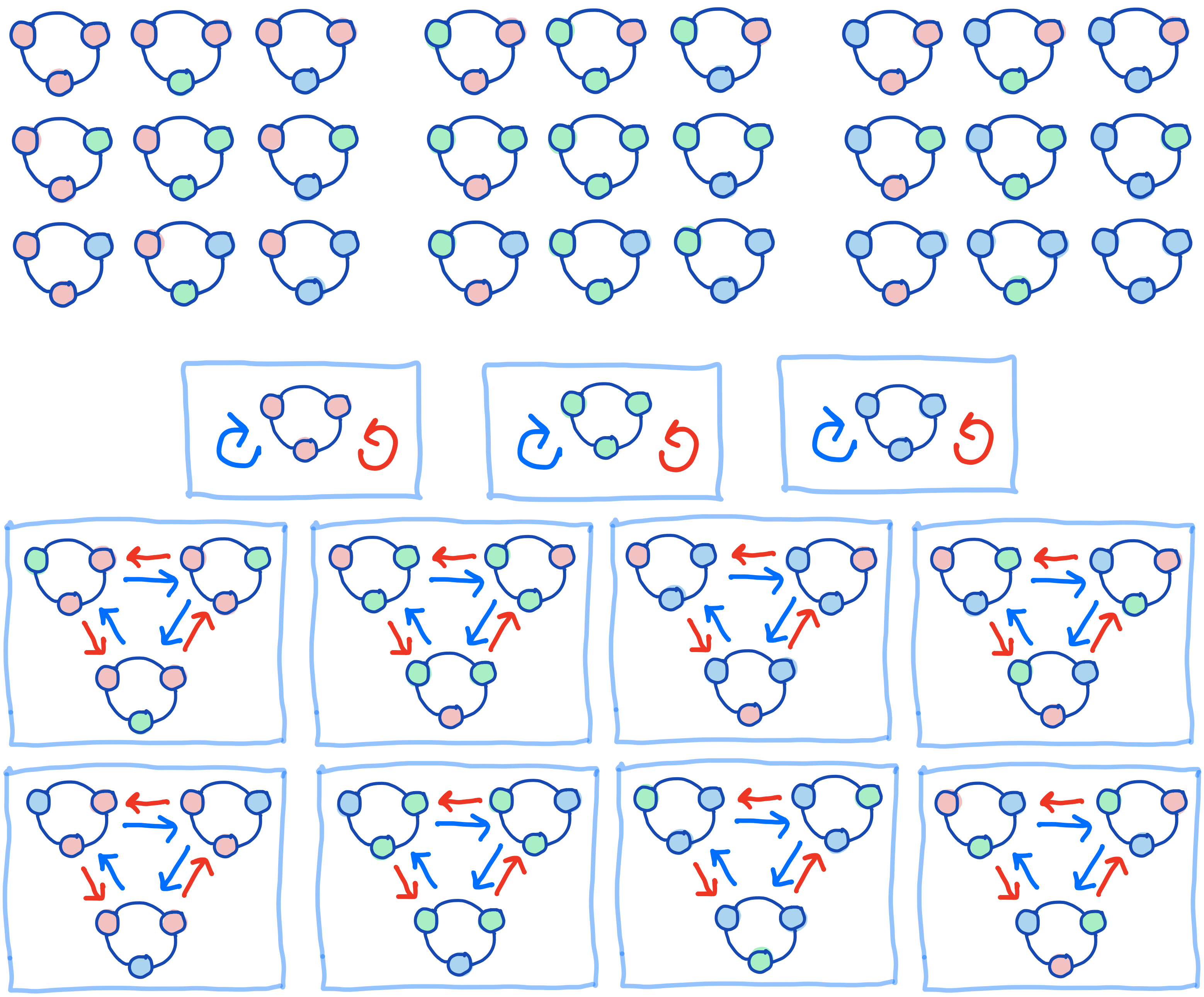


" $n^{\lceil \frac{k}{2} \rceil}$ counting problem

Divisibility = more symmetry

$n = K = 3$ $\quad$ $G = \{\ \boxed{1}, \boxed{\circlearrowright}, \boxed{\circlearrowleft}\ \}$

do nothing $\quad$ 1/3 turn → $\quad$ 1/3 turn →



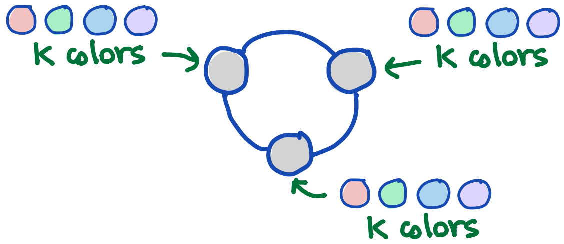
$|G| = 3 \qquad |X| = 27 = |X_1| \qquad |X_{\circlearrowright}| = |X_{\circlearrowleft}| = 3$

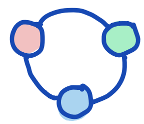
$$\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{3}\left(|X_1| + |X_{\circlearrowright}| + |X_{\circlearrowleft}|\right) = \frac{1}{3}(27+3+3) = 11$$ ☑

$n = 3$ any k $G = \{\ 1,\ \circlearrowright,\ \circlearrowleft\ \}$

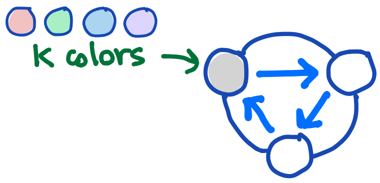
do nothing, $\frac{1}{3}$ turn →, $\frac{1}{3}$ turn →

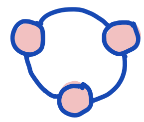
$|X| = |X_1| = k^3$



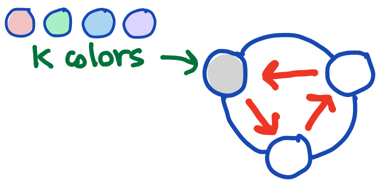


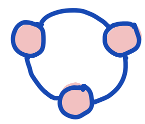
$|X_{\circlearrowright}| = k$





$|X_{\circlearrowleft}| = k$



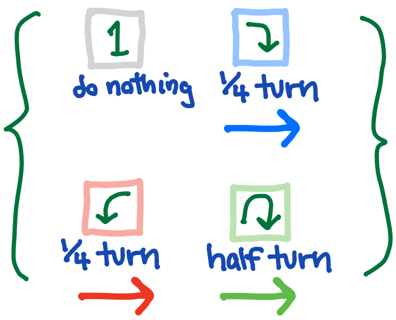


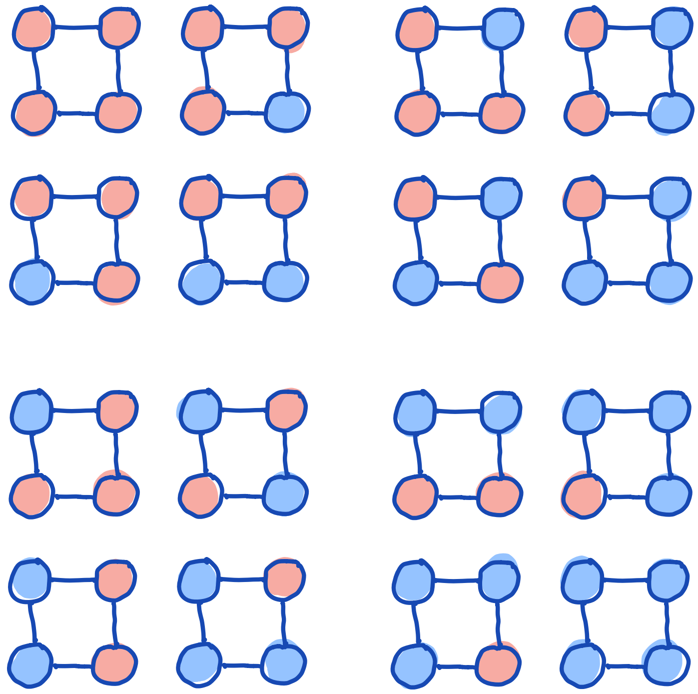
$$\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{3}\left(|X_1| + |X_{\circlearrowright}| + |X_{\circlearrowleft}|\right) = \frac{1}{3}\left(k^3 + k + k\right)$$

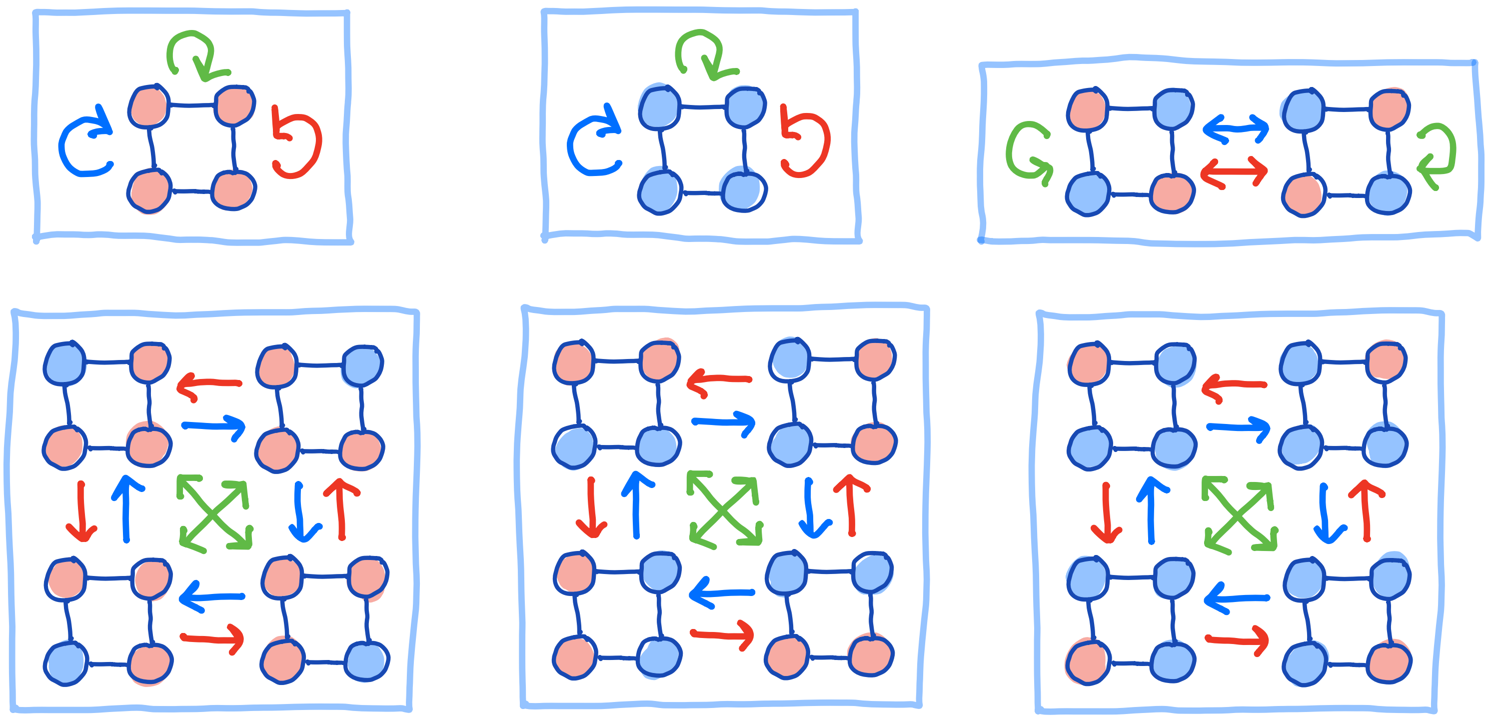
Check: $k=3$ $\frac{1}{3}\left(k^3+k+k\right) = \frac{1}{3}(27+3+3) = 11$ ☑

$n=4 \quad k=2$

$G =$





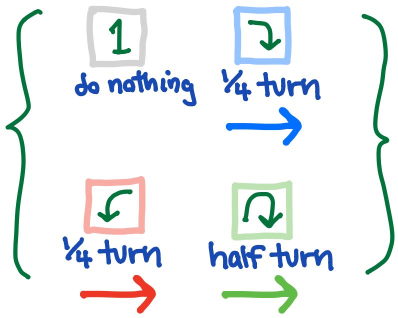


$|G| = 4 \qquad |X| = 16 = |X_1| \qquad |X_{\curvearrowright}| = |X_{\curvearrowleft}| = 2 \qquad |X_{\circlearrowright}| = 4$

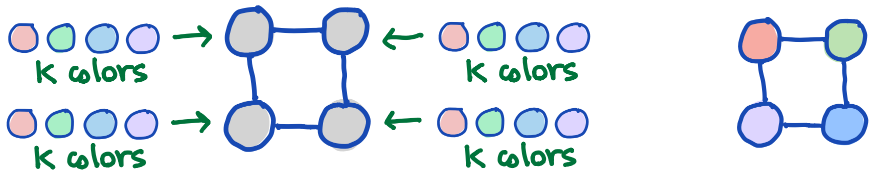
$$\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{4}\left(|X_1|+|X_{\curvearrowright}|+|X_{\curvearrowleft}|+|X_{\circlearrowright}|\right) = \frac{1}{4}(16+2+2+4) = 6$$

☑

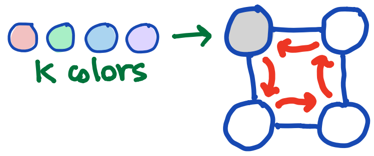
$n = 4$ any k

$G =$ 

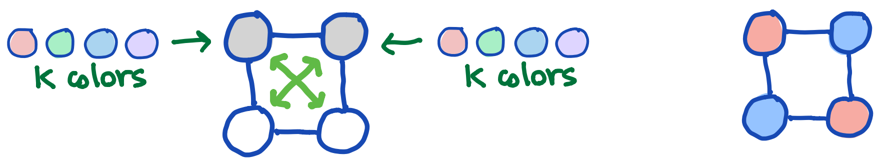
$|X| = |X_1| = k^4$



$|X_{\curvearrowright}| = |X_{\curvearrowleft}| = k$



$|X_{\circlearrowright}| = k^2$

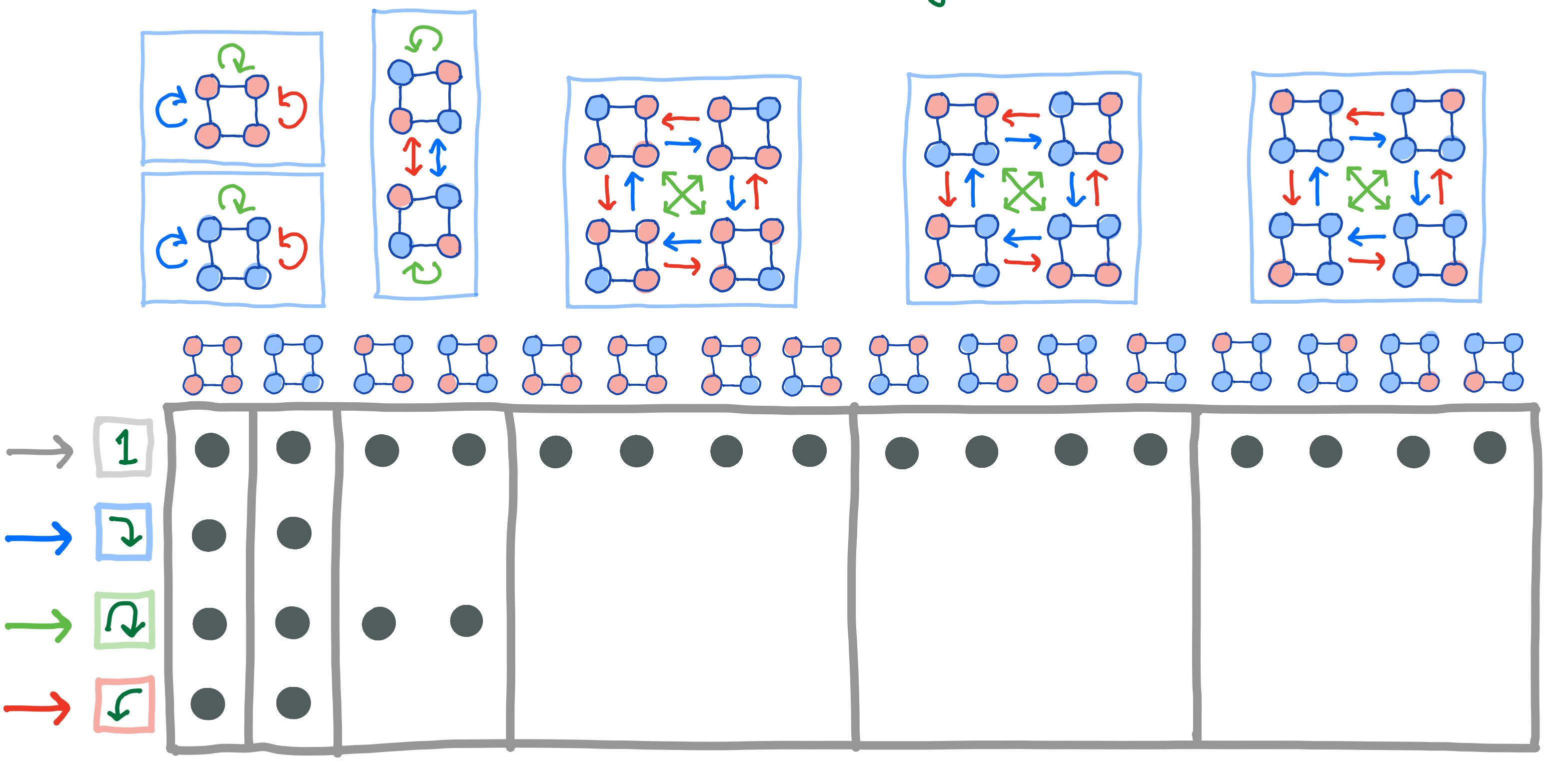


$$\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{4}\left(|X_1| + |X_{\curvearrowright}| + |X_{\curvearrowleft}| + |X_{\circlearrowright}|\right) = \frac{1}{4}\left(k^4 + k + k + k^2\right)$$

Check: $k = 2$ $\quad \frac{1}{4}\left(k^4 + k + k + k^2\right) = \frac{1}{4}(16 + 2 + 2 + 4) = 6$ ☑

Why does this work?

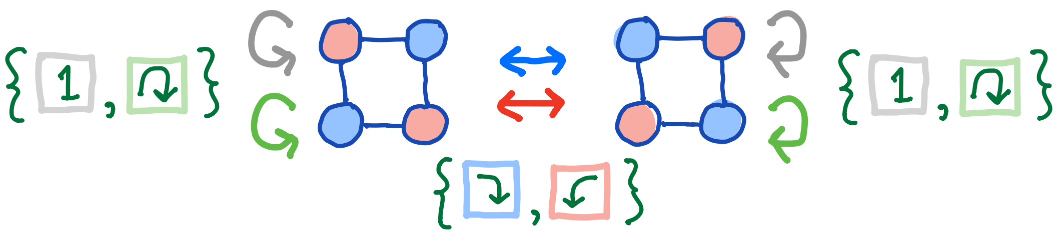
$$\frac{1}{|G|}\sum_{g\in G}|X_g| = |M|$$

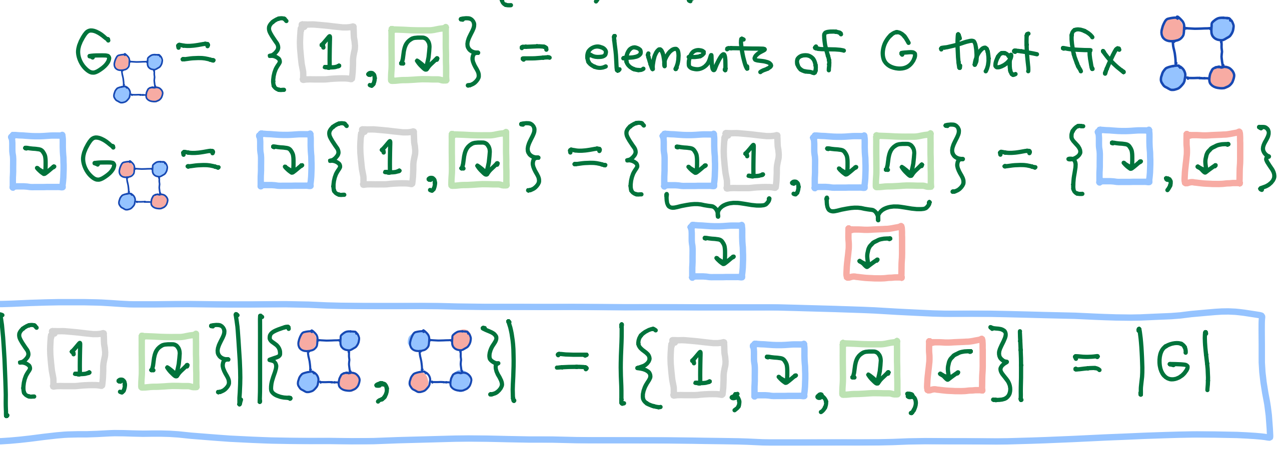


Each dot ● marks an object fixed b a group element.
Each box is a pattern up to symmetry.
The row sums are $|X_1|$, $|X_{↷}|$, $|X_{↷↷}|$, $|X_{↶}|$.
If we can figure out why each box gets |G| dots, we're done.

Group Theory in a nutshell: things divide up evenly.

Look more closely at each orbit. This one is interesting:





Combinatorics Feb23

What is a group? One operation $*$ or $+$
Identity and inverses
Associative: $(ab)c = a(bc)$

$\mathbb{Z}_2$:

+	0	1
0	0	1
1	1	0

mod 2

≈

+	even	odd
even	even	odd
odd	odd	even

≈

*	1	-1
1	1	-1
-1	-1	1

≈

*	1	2
1	1	2
2	2	1

mod 3

$\mathbb{Z}_3$:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

mod 3

$\mathbb{Z}_4$:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

mod 4

≈

*	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

mod 5

+		*
0	↔	1
1	↔	2
2	↔	4
3	↔	3

$\mathbb{Z}_2 \times \mathbb{Z}_2$:

+	0,0	0,1	1,0	1,1
0,0	0,0	0,1	1,0	1,1
0,1	0,1	0,0	1,1	1,0
1,0	1,0	1,1	0,0	0,1
1,1	1,1	1,0	0,1	0,0

mod 2,2

$\mathbb{Z}_5$:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\mathbb{Z}_6$:

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

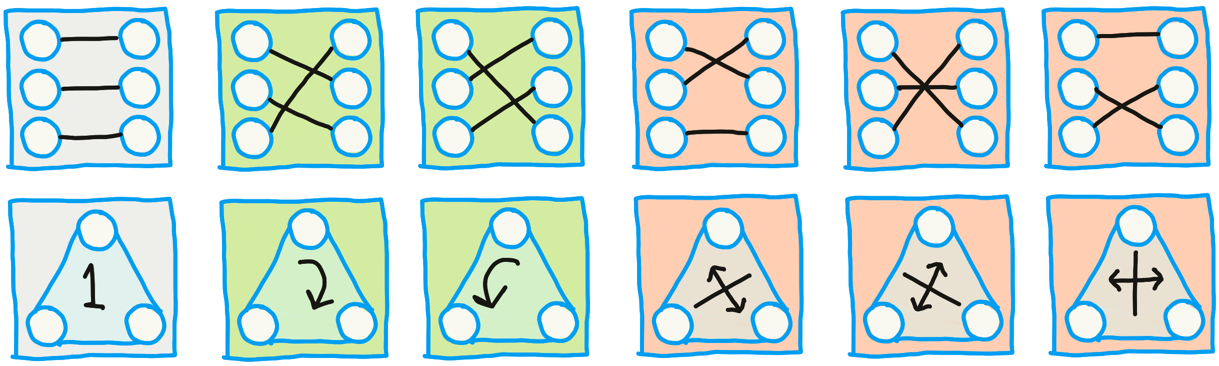
$\mathbb{Z}_2 \times \mathbb{Z}_3$:

+	0,0	0,1	0,2	1,0	1,1	1,2
0,0	0,0	0,1	0,2	1,0	1,1	1,2
0,1	0,1	0,2	0,0	1,1	1,2	1,0
0,2	0,2	0,0	0,1	1,2	1,0	1,1
1,0	1,0	1,1	1,2	0,0	0,1	0,2
1,1	1,1	1,2	1,0	0,1	0,2	0,0
1,2	1,2	1,0	1,1	0,2	0,0	0,1

mod 2,3

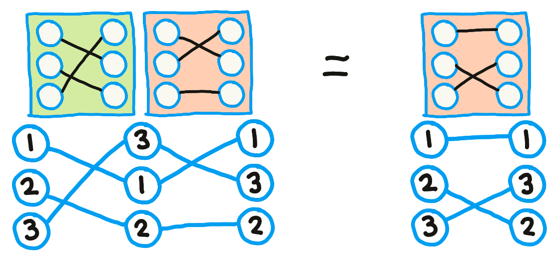
Inverses ⟺ Each row is a permutation of the first row
Each col is a permutation of the first col

The symmetric group S_3: Permutations of $\{1,2,3\}$
Symmetries of a triangle

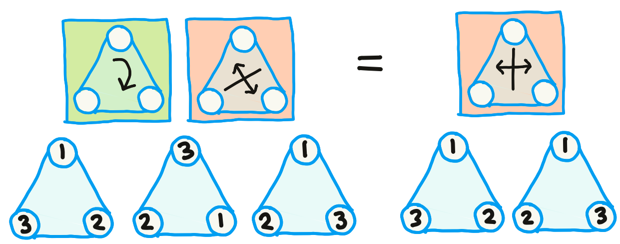


How to multiply?

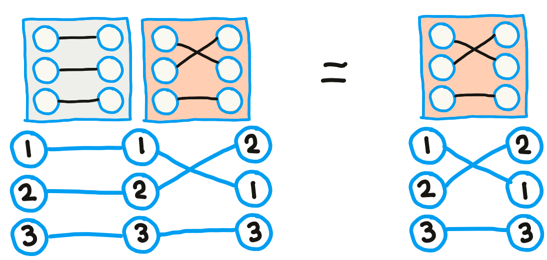
→ Pull tight



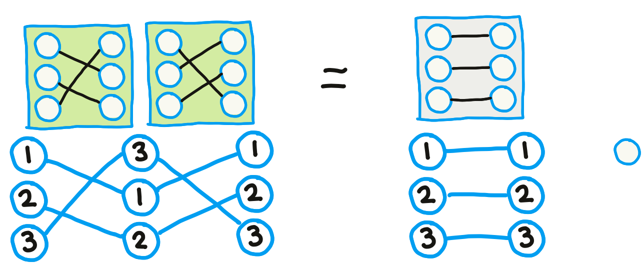
→ Watch test triangle



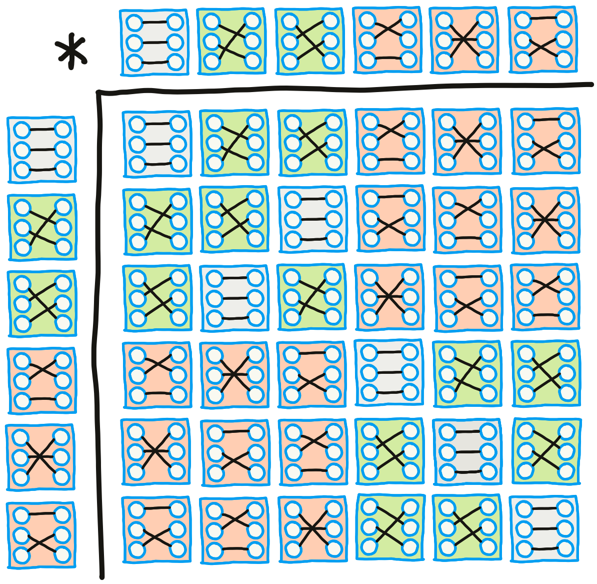
Identity

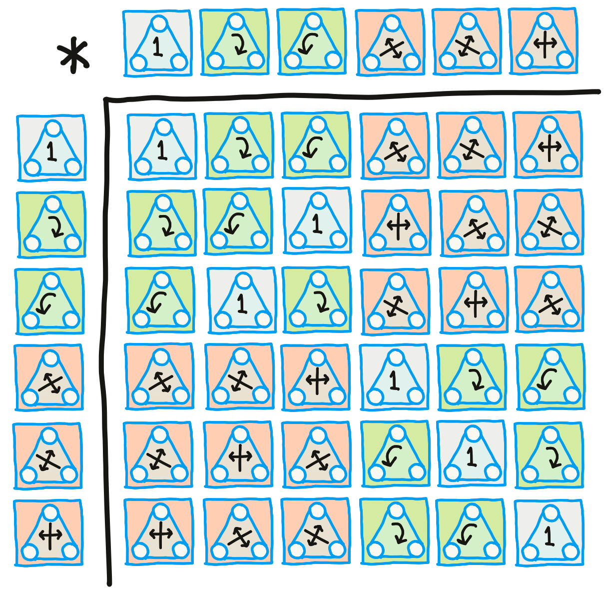


Inverses

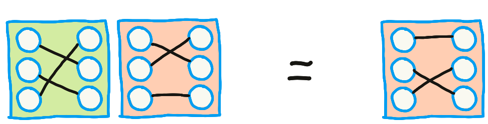


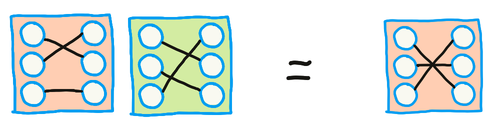
S_3 multiplication tables





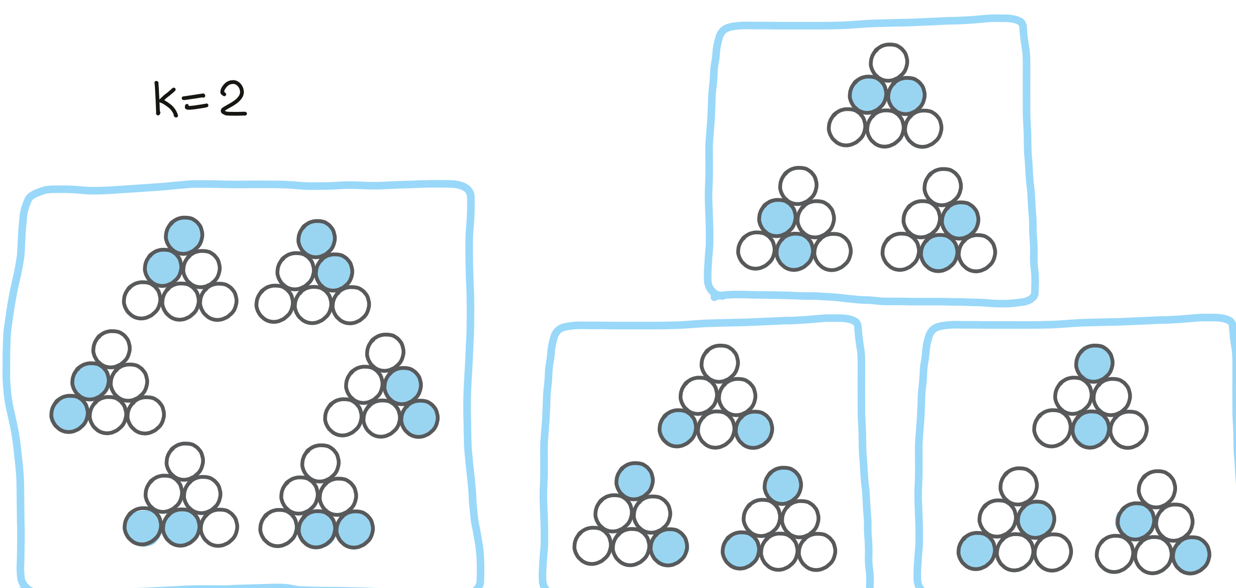
Not commutative



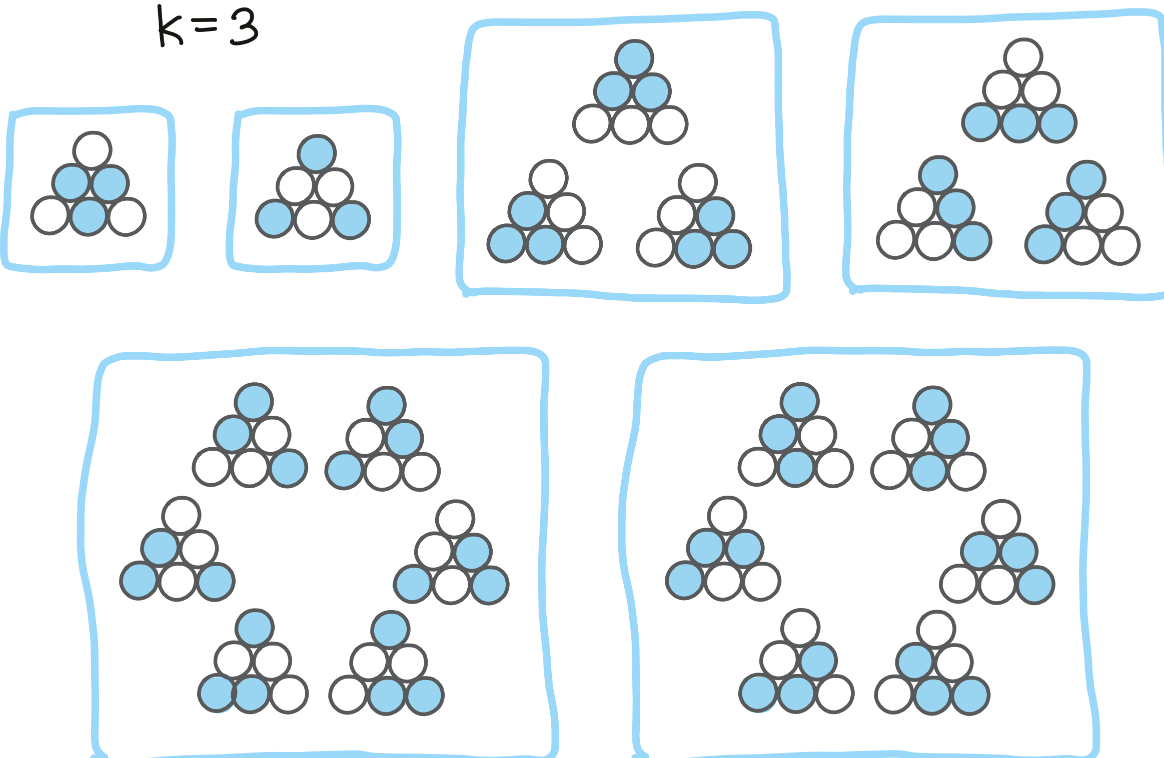


Counting problem: Mark k cells in a triangular grid
How many patterns, up to S_3 symmetry?

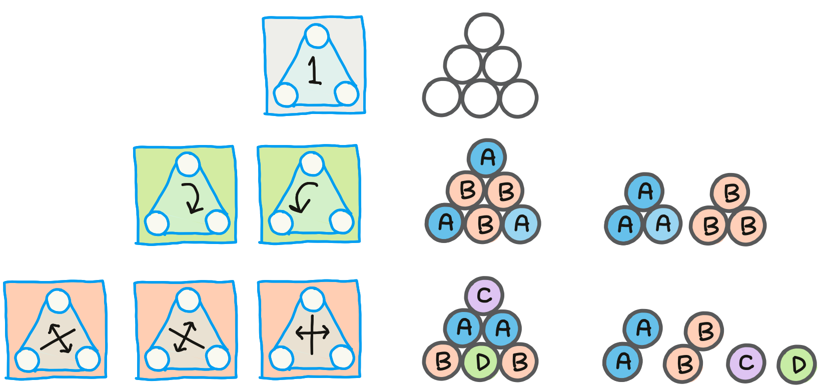
$k=2$



k=3



$$|P| = \frac{1}{|G|}\sum_{g\in G}|X_g|$$

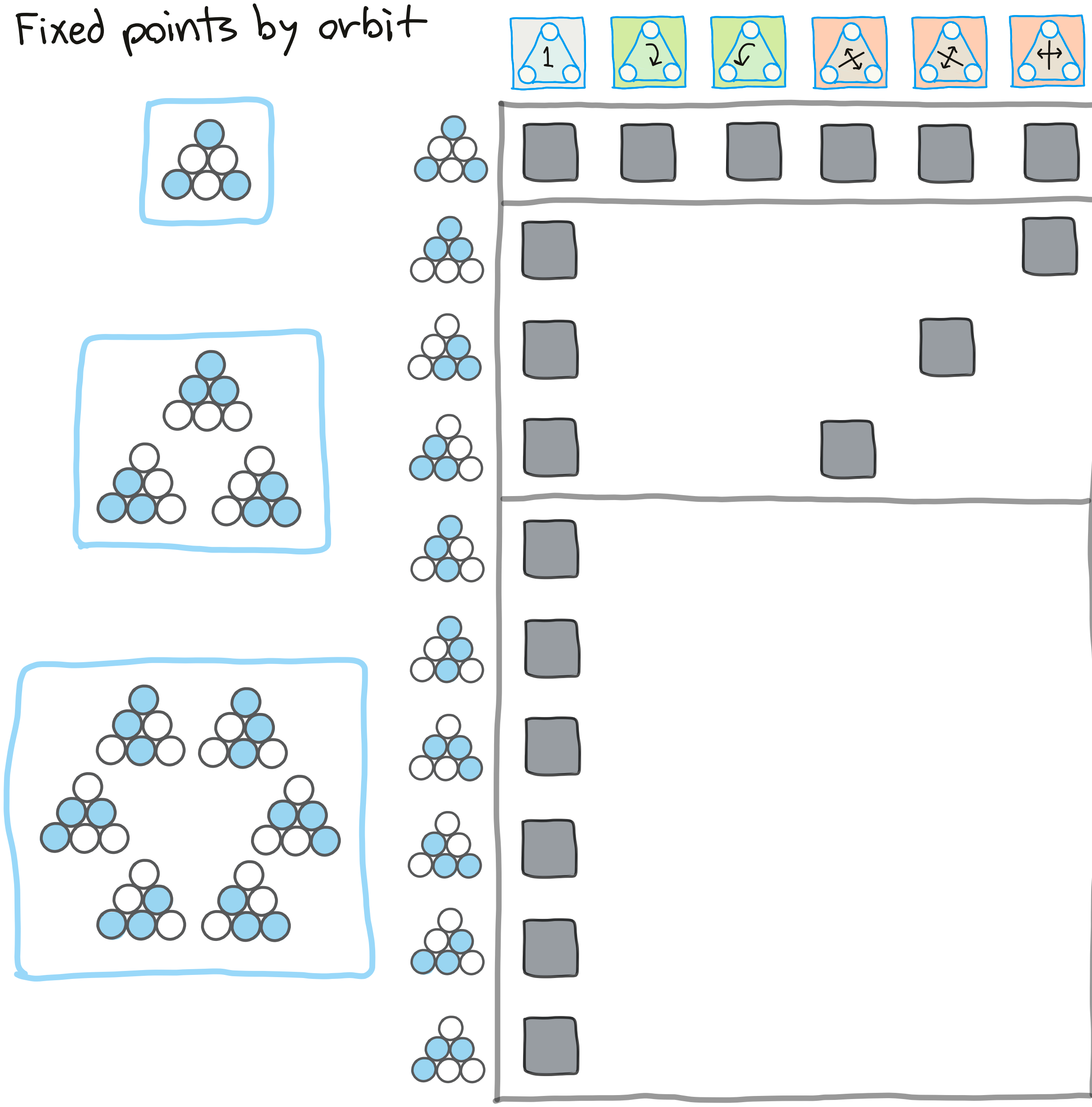


k=2	k=3
$\binom{6}{2}$	$\binom{6}{3}$
0	$\binom{2}{1}$
$\binom{2}{1}+\binom{2}{2}$	$\binom{2}{1}\binom{2}{1}$

k=2: $\frac{1}{6}\left[\binom{6}{2}+3\left(\binom{2}{1}+\binom{2}{2}\right)\right] = \frac{1}{6}(15+3\cdot3) = 4$ ☑

k=3: $\frac{1}{6}\left[\binom{6}{3}+2\binom{2}{1}+3\binom{2}{1}\binom{2}{1}\right] = \frac{1}{6}(20+2\cdot2+3\cdot4) = 6$ ☑

Fixed points by orbit

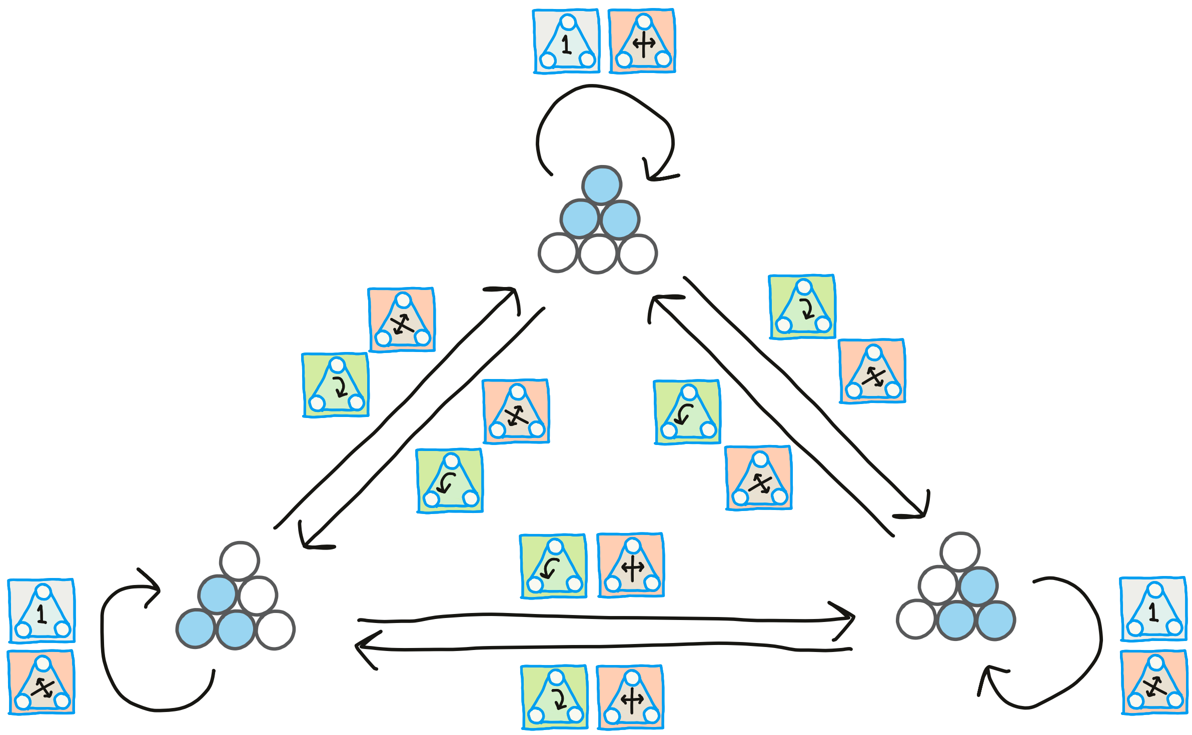


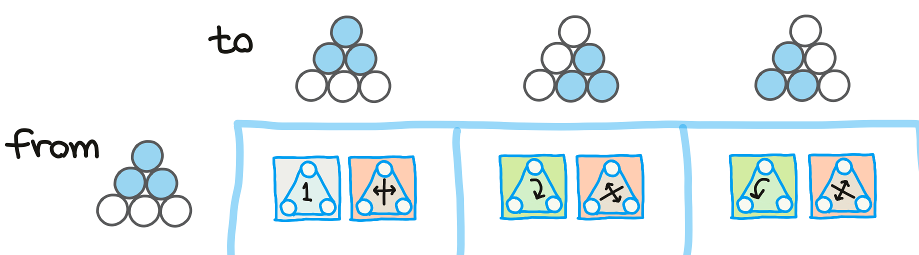
$\sum_{g \in G} |X_g|$ counts all fixed points (g,x) where $gx = x$

If we can understand why there are $|G|$ fixed points per orbit,

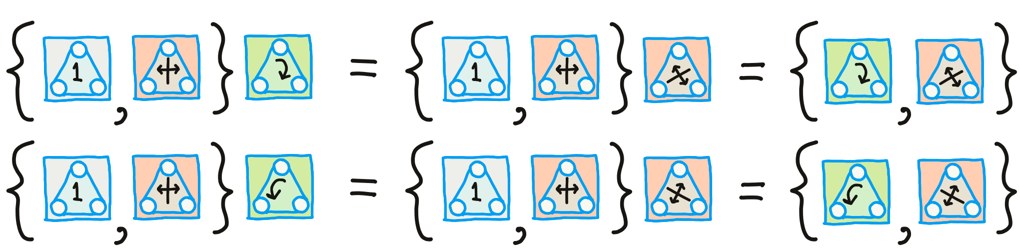
then we understand $|P| = \frac{1}{|G|} \sum_{g \in G} |X_g|$

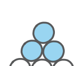
Look closely at how G acts on a particular orbit

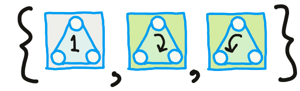


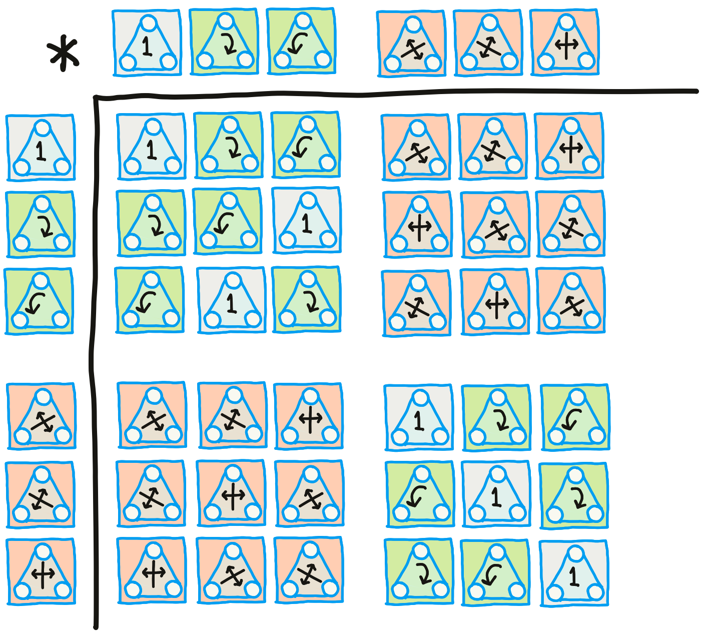


These subsets of G (cosets) are always in 1:1 correspondence with each other, so they divide G into equal sized subsets.



(# Fixed points of )(size of orbit) = |G|

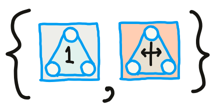
Quotients: mod out by "normal subgroup" 



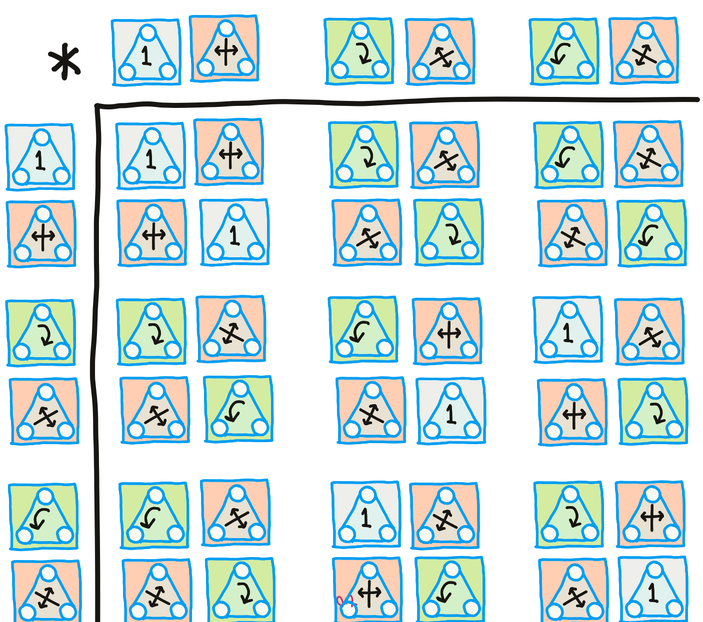
⟹

+	0	1
0	0	1
1	1	0

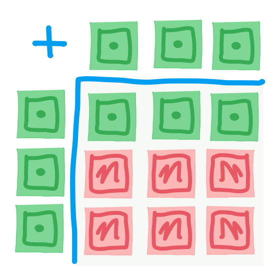
mod 2

 is not normal

and we don't get a coherent table when we try to mod out.



⟹

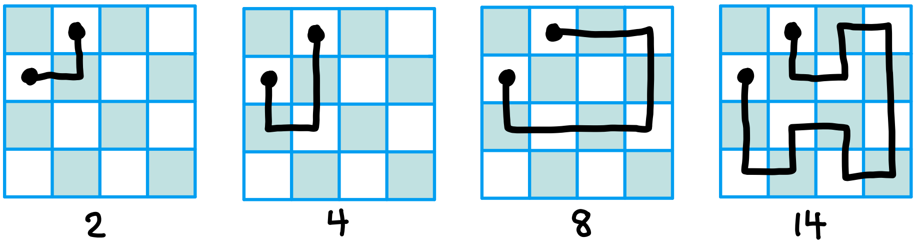


Various entries are inconsistent

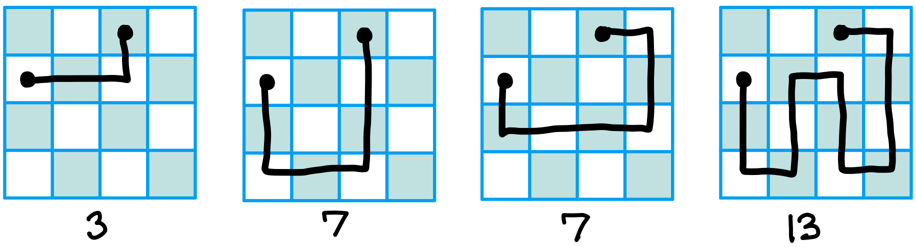
Expand on class questions:

Even-odd parity.

Walks alternate square colors



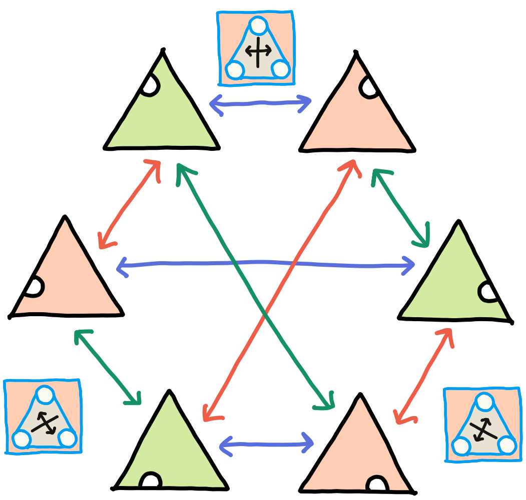
Walks between squares of the same color: even # steps

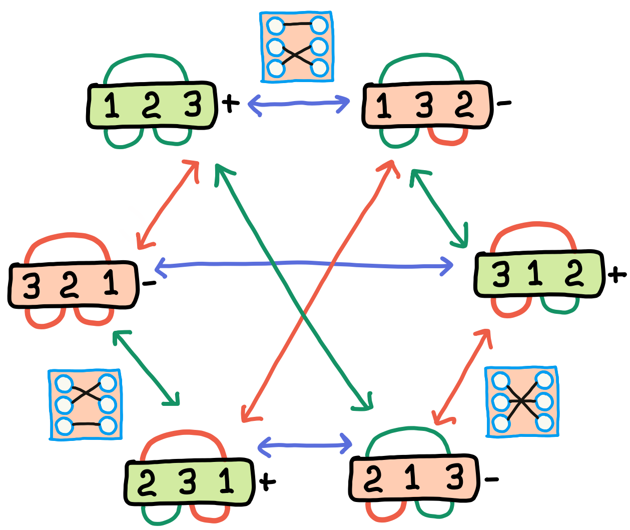


Walks between squares of the opposite color: odd # steps

We can checkerboard the graph of all triangle positions.

Flips all change checkerboard color

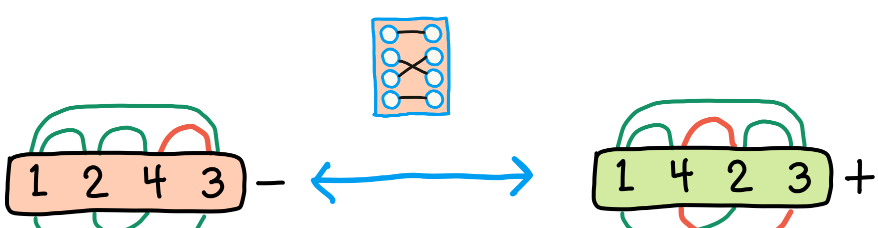




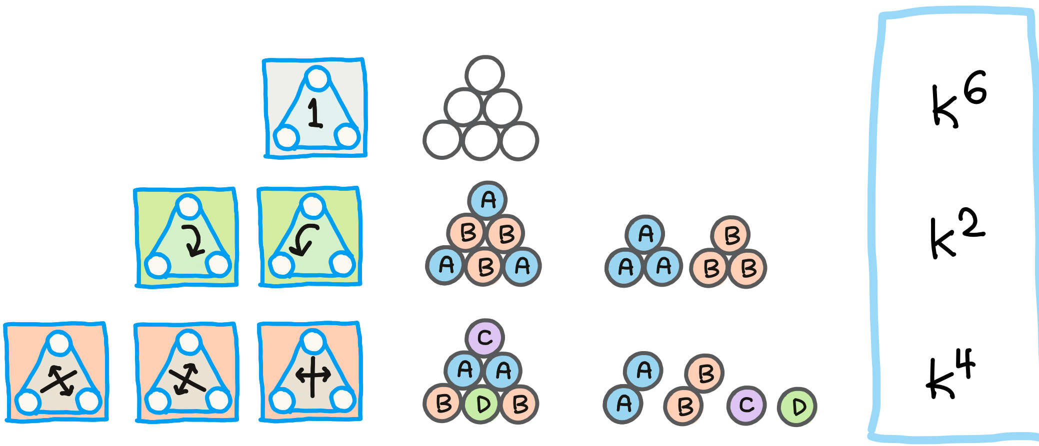
We can checkerboard the graph of all permutations of $\{1, \ldots, n\}$

Even-odd: How many pairs are out of order?

Adjacent pair swaps change this count by 1



k colors $\quad |P| = \frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{6}(k^6+2k^2+3k^4)$

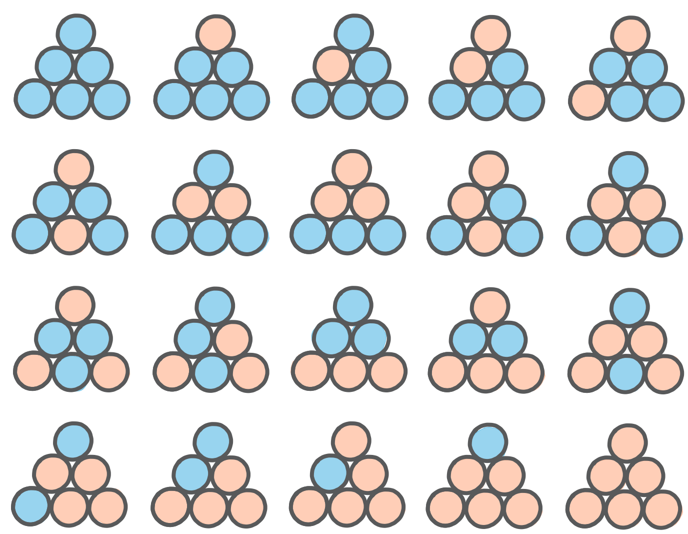


$k=2$

$$|P| = \frac{1}{6}(k^6+2k^2+3k^4)$$

$$= \frac{1}{6}(64+\underbrace{2\cdot 4+3}_{12}\cdot\underbrace{16}_{8})$$

$$= 20$$



$k=3 \quad |P| = \frac{1}{6}(k^6+2k^2+3k^4) = \frac{1}{6}(729+2\cdot 9+3\cdot 81) = 165$

use 1 color: 3

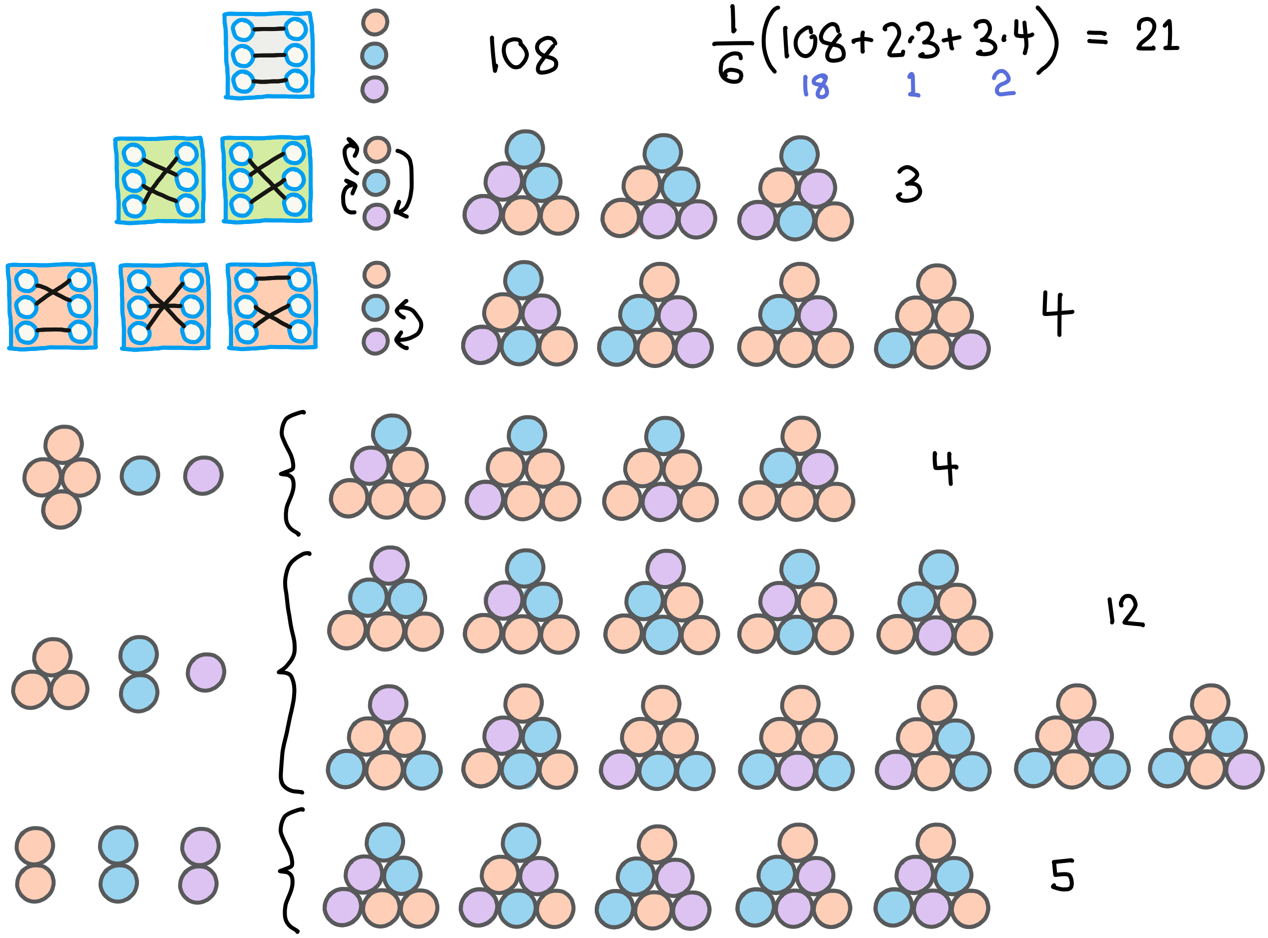
use 2 colors: $\binom{3}{2}18$ (from above)

$\Rightarrow$ use 3 colors: $165-3-\binom{3}{2}18 = 108$

Not easily checked

(This way lies madness)

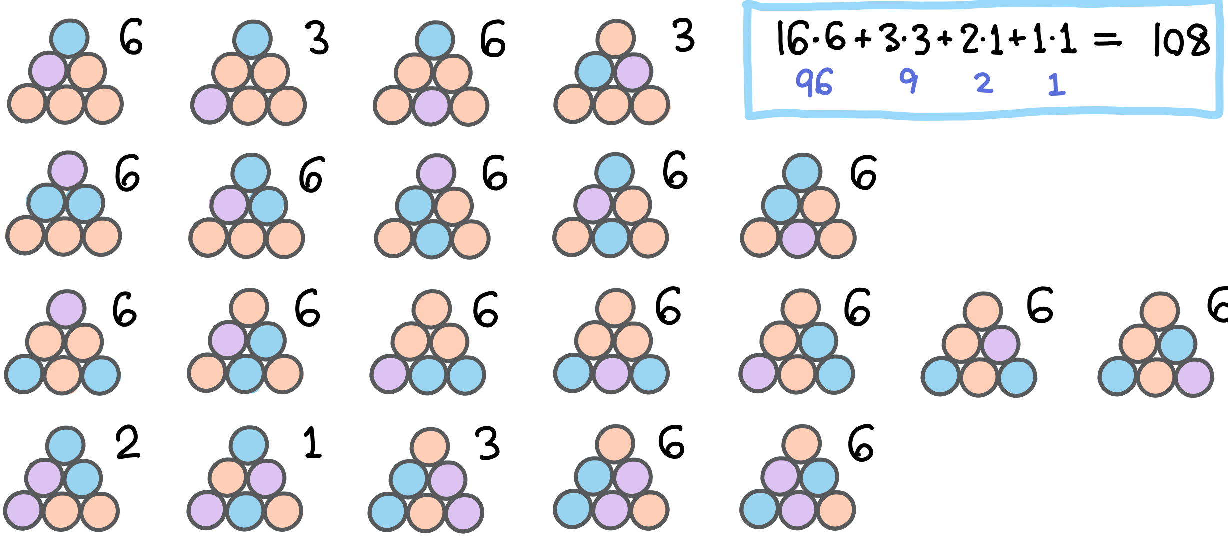
Let S_3 act on the colors, for this $|X| = 108$



$$\frac{1}{6}\left(108 + 2\cdot3 + 3\cdot4\right) = 21$$

18   1   2

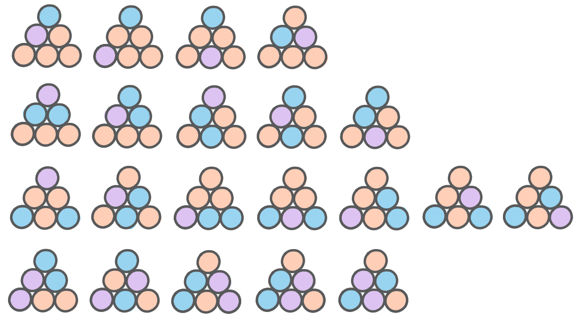
Now count orbit sizes by S_3 acting on colors

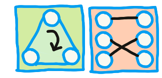


$$16\cdot6 + 3\cdot3 + 2\cdot1 + 1\cdot1 = 108$$

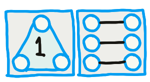
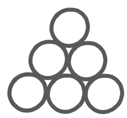
96   9   2   1

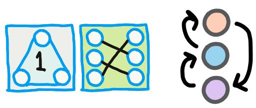
More systematic way to get 21 ways to color  using 3 interchangeable colors up to triangle symmetries:



Let $G = S_3 \times S_3$, group of pairs of actions of form  acting on triangle and then color choices

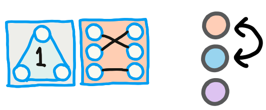
$|G| = |S_3||S_3| = 6 \cdot 6 = 36$

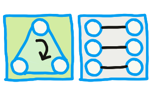
15 1   $3^6 - 3 \cdot 2^6 + 3 = 729 - 192 + 3 = 540$

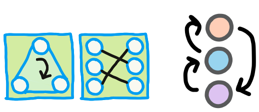
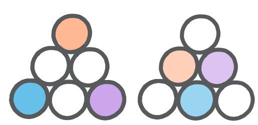
2  none

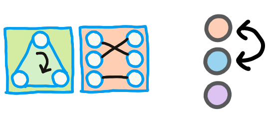
$\frac{1}{36}(540 + 4 \cdot 9 + 3 \cdot 36 + 9 \cdot 8) = 21$ ☑

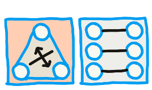
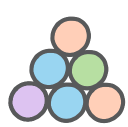
15 1 3 2

3  none

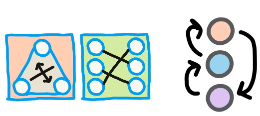
2  none

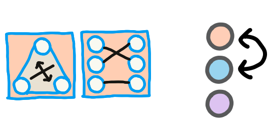
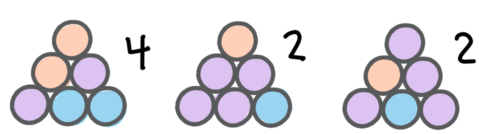
1 4   $3 \cdot 3 = 9$

6  none

3 3   4 zones color using all 3 colors

$3^4 - 3 \cdot 2^4 + 3 = 81 - 48 + 3 = 36$

6  none

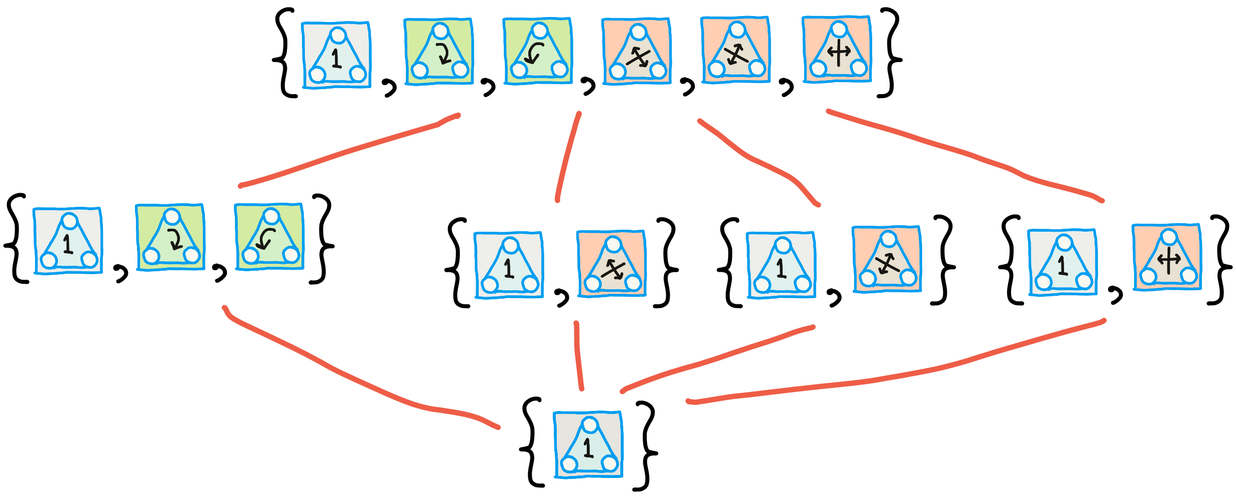
2 9  

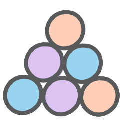
8

21 36

☑ ☑

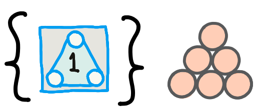
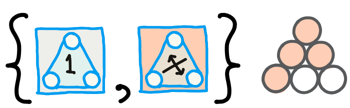
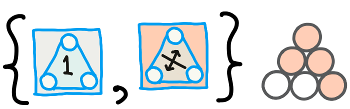
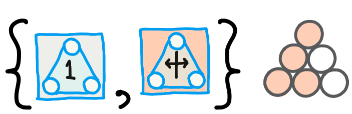
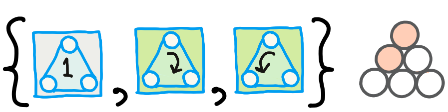
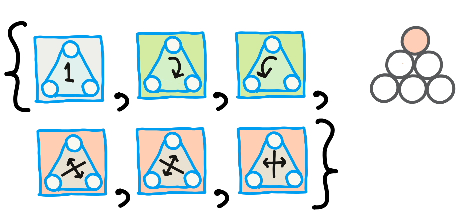
Can we use inclusion-exclusion instead of Burnside's lemma?
Need to consider poset of subgroups of S_3. Möbius inversion.





k colors

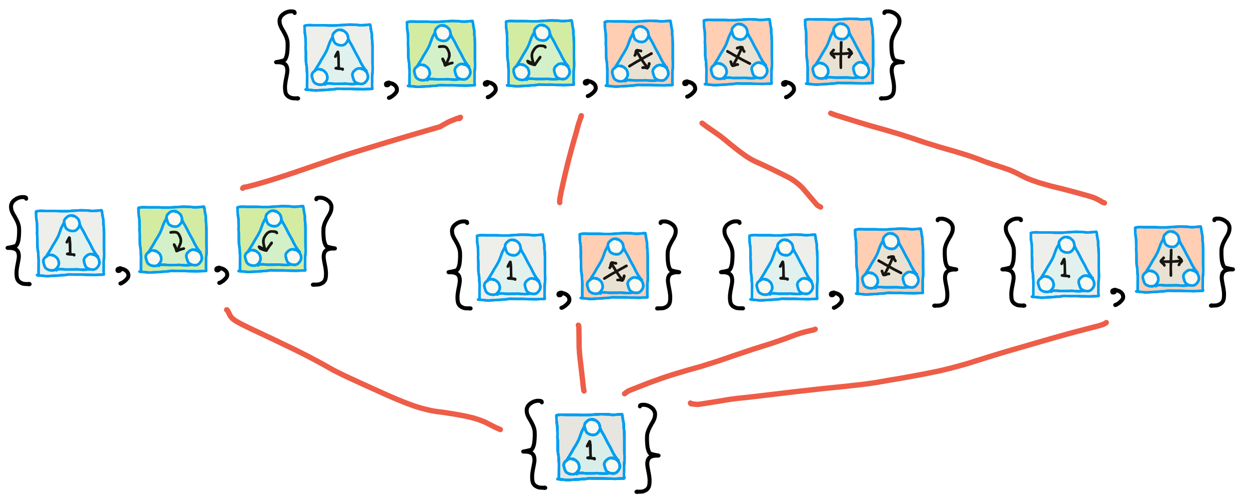
$$|P| = \frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{6}(k^6+2k^2+3k^4)$$

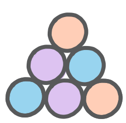
type of symmetry	at least	exactly, divided by symmetries	k^6	k^4	k^2	k
	k^6	$\frac{1}{6}(k^6-3k^4-k^2+3k)$	1	-3	-1	3
	k^4	$\frac{1}{3}(k^4-k)$		2		-2
	k^4	$\frac{1}{3}(k^4-k)$		2		-2
	k^4	$\frac{1}{3}(k^4-k)$		2		-2
	k^2	$\frac{1}{2}(k^2-k)$			3	-3
	k	k				6
			1	3	2	0

$(k^6\ k^4\ k^2\ k)/6$

$\frac{1}{6}(k^6+2k^2+3k^4)$ ☑

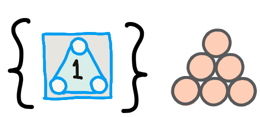
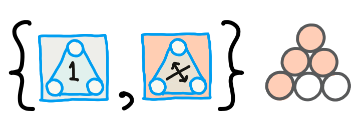
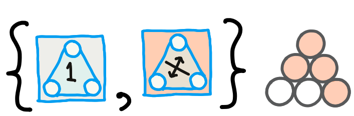
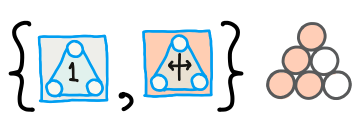
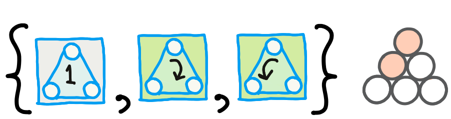
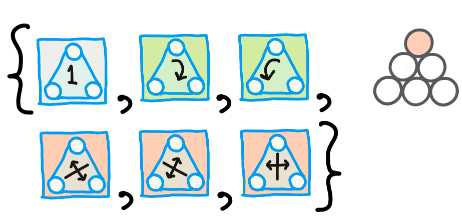
Better approach: Skip Möbius inversion to compute "exactly".
Rather, when a pattern has d versions, we want to count each one with weight $1/d$.
Work up the poset, adjusting weights based on count so far from below.





k colors

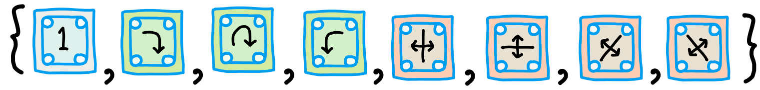
$$|P| = \frac{1}{|G|} \sum_{g \in G} |X_g| = \frac{1}{6}(k^6 + 2k^2 + 3k^4)$$

type of symmetry	at least	desired weight	subtract below	net contribution
	k^6	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}k^6$
	k^4	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}k^4$
	k^4	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}k^4$
	k^4	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}k^4$
	k^2	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}k^2$
	k	1	0	

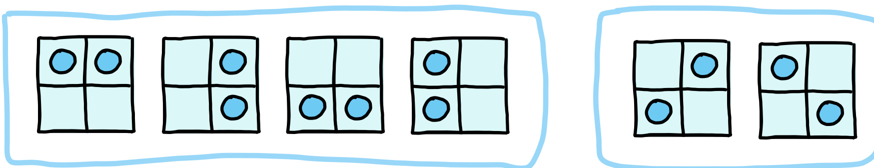
$\frac{1}{6}(k^6 + 2k^2 + 3k^4)$ ☑

This can be easier than Burnside's lemma.

Placing k markers on an n×n board, up to symmetry.

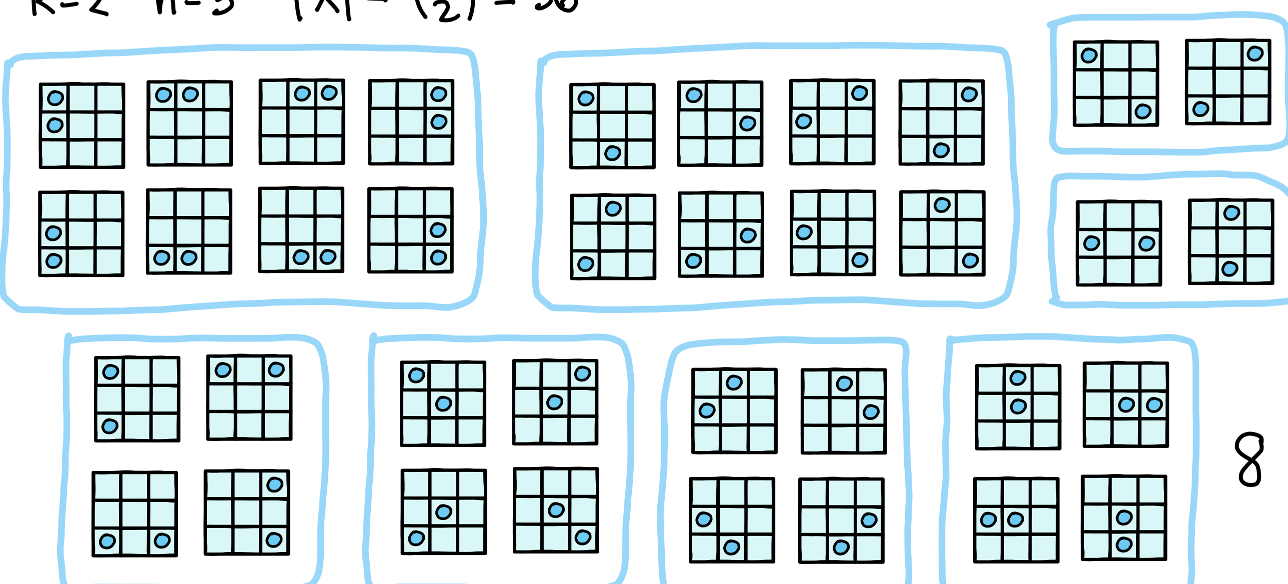
$G = \{$  $\}$ $\quad |G| = 8$

$k=n=2 \quad |X| = \binom{4}{2} = 6$

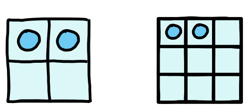


2

$K=2 \quad n=3 \quad |X| = \binom{9}{2} = 36$

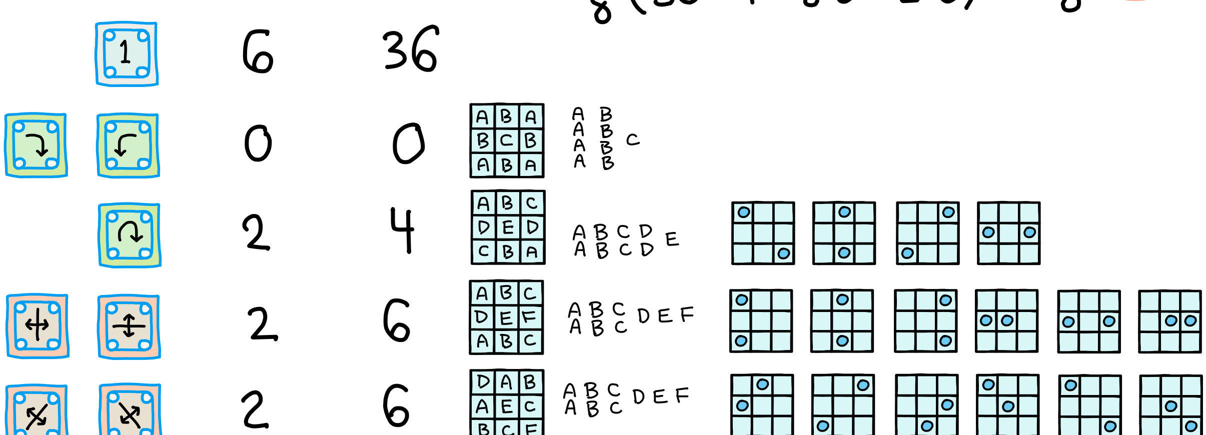


8



$\frac{1}{8}(6 + 2 + 2\cdot 2 + 2\cdot 2) = 2$ ☑

$\frac{1}{8}(36 + 4 + 2\cdot 6 + 2\cdot 6) = 8$ ☑



March 9, 2021 — Counting with symmetries on polytopes.

Symmetries of space

Linear Algebra — w/o angle, length; then add these <fig> f.g

xkcd.com

chirality

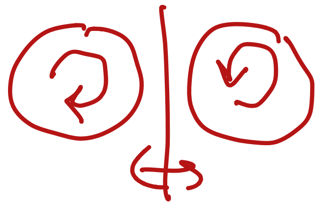
orthonormal basis $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$

$v_1 \perp v_2$, $v_1 \perp v_3$, $v_2 \perp v_3$ — $|v_i| = 1$, $v_i \cdot v_i = 1$

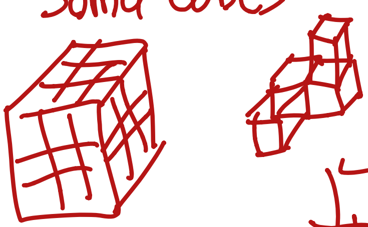
$v_i \perp v_j = v_i \cdot v_j = 0$

$$\underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A^{-1} = A^T$ $\det(A) = 1$

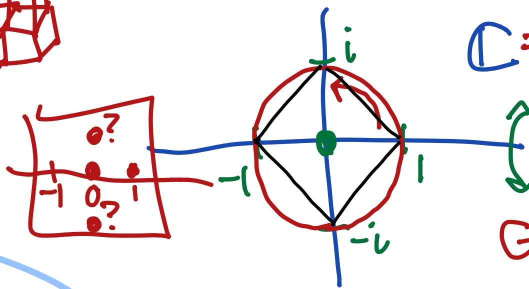


Soma cubes



rotations in space

$\mathbb{C} = \mathbb{R}^2$



$\overline{a+bi} = a - bi$

$\bar{i} = -i$

$G = \{c \in \mathbb{C} \mid |c| = 1\} = SO(2)$

c·d rotations

$x^2 + 1 = 0$

$(x+i)(x-i) = 0$

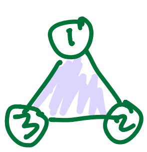
$O(n)$ = orthonormal matrices $\mathbb{R}^n \to \mathbb{R}^n$

$SO(n) = \ldots \det = 1$ rotations

n-simplex

- interval •—• 1-simplex
- triangle △ 2-simplex
- tetrahedron 3-simplex

symmetric group: permutations of $\{1, \ldots, n\}$ — S_n all, A_n even

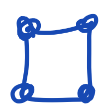


geometric view

permutation view

$|S_4| = 24$

$|A_4| = 12$



$|S_4| = 24 = 4!$

$|G| = 8$

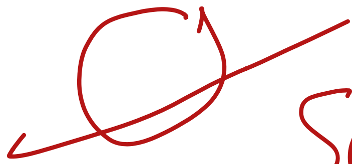
S_5

"A_5 can't be factored" A_n $n \geq 5$

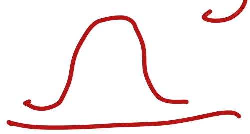
$$ax^2+bx+c=0 \Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$\sqrt{}$

$\overline{a+bi}$
$=a-bi$



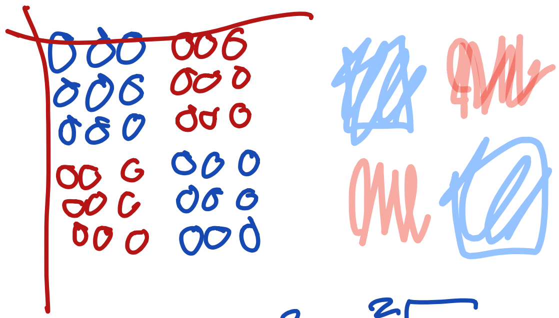
$\int e^{x^2}dx$



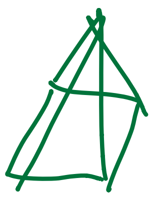
erf(x)

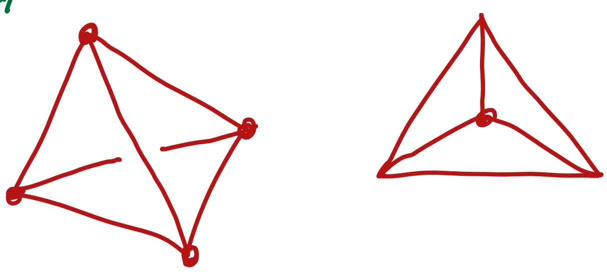
$\sqrt{}$ $\sqrt[3]{}$

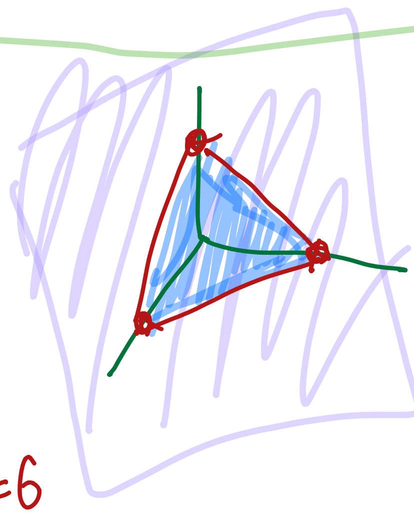
$\sqrt[4]{}$



2 $\sqrt[2]{}$



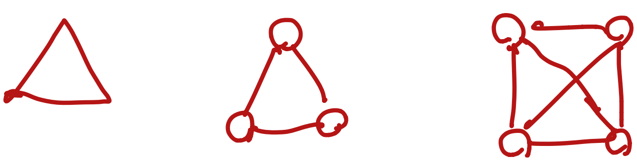




all $v=(x,y,z)$

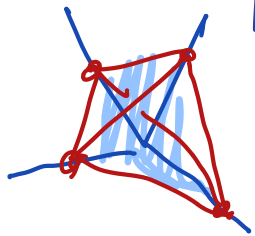
$x,y,z \geq 0$

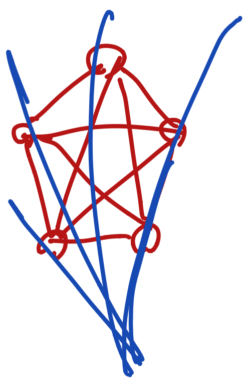
$x+y+z=1$



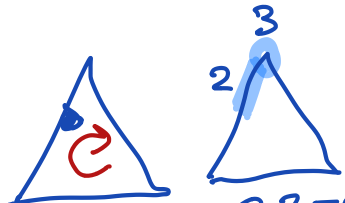
$\binom{4}{2}=6$

$\mathbb{R}^4$

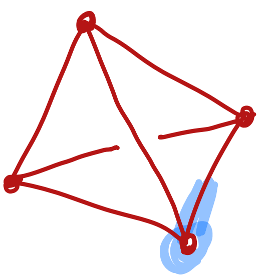




$\binom{5}{2}=10$



$3\cdot 2=6$



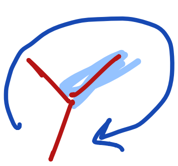
mark it to destroy symmetry

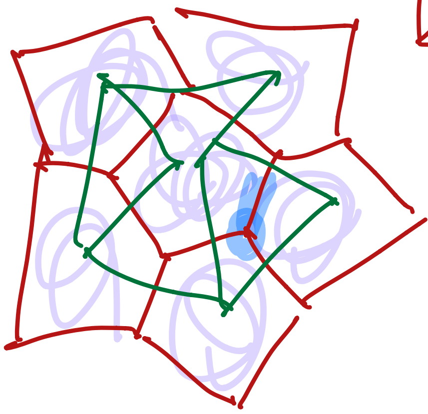
count choices

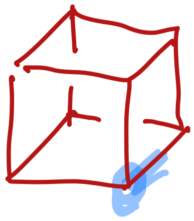
4 choices of corner

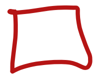
× 3 choices of edge meeting that corner

12









$|G| = 8$

$|S_4| = 24$

$|G| = 8 \cdot 3 = 24$

$|S_8| = 8!$

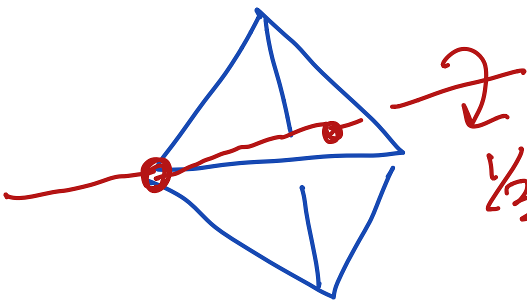
$12 \cdot 5/3 = 20$

$|G| = 20 \cdot 3 = 6$

$G = A_5$

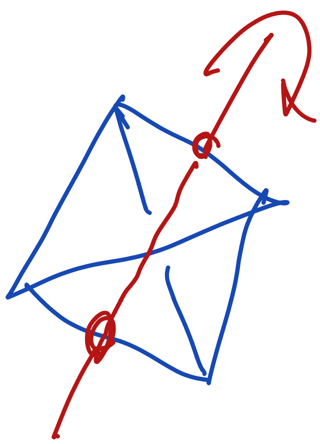
\# ways k-color faces of a tetrahedron up to symmetry

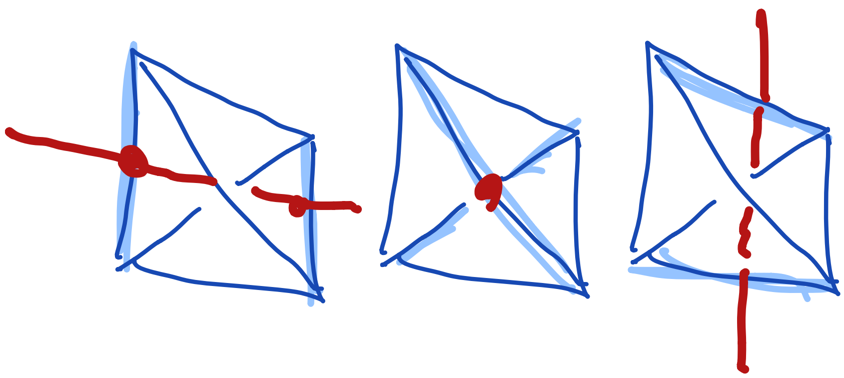
$|G| = 12$

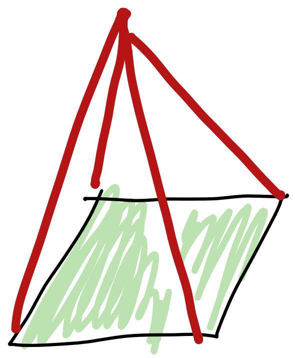


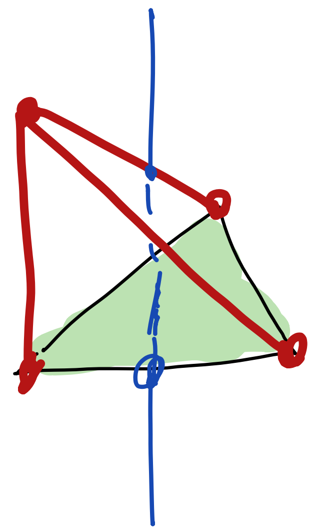
1/3 turn

1	do nothing, identity
8	1/3 turns (4 vertices x2 turns)
3	1/2 turns
12	

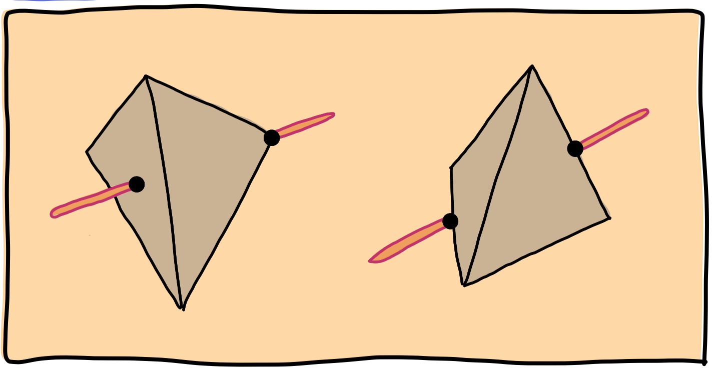


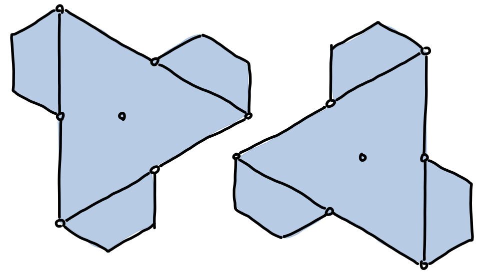






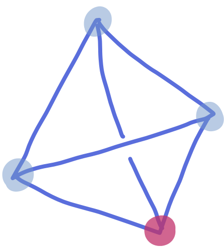
March 11



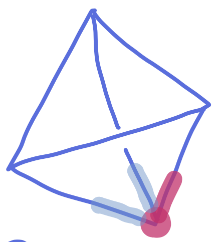


I have posted plans for the above model on our website.

The tetrahedron has 12 symmetries:



① Choose a corner
4 choices



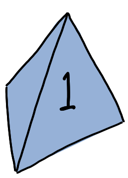
② choose an edge
meeting that corner
3 choices

G = group of symmetries of tetrahedron in $\mathbb{R}^3$
(we ignore flips through $\mathbb{R}^4$)

$|G| = 4 \cdot 3 \quad 12$

Can we find these 12 symmetries?

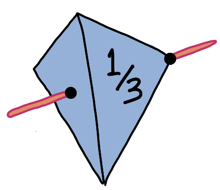
1



Identity

Do nothing

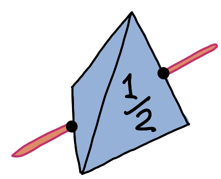
$2 \cdot 4$



1/3 turn either way

axis through
face and vertex

3



1/2 turn

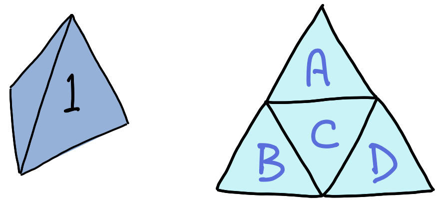
axis through
opposite edges

$1 + 2 \cdot 4 + 3 = 12$ ☑

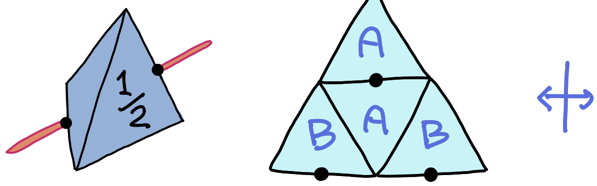
Burnside's lemma:

$$\frac{1}{|G|} \sum_{g \in G} |X_g|$$

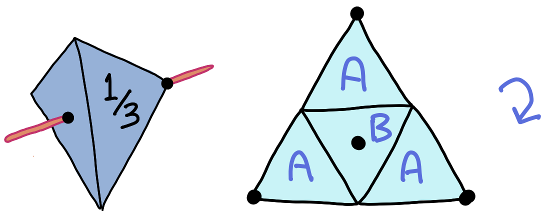
Example: How many ways can we color the sides of a tetrahedron, up to symmetry, using k colors?

1 

k^4

3 

k^2

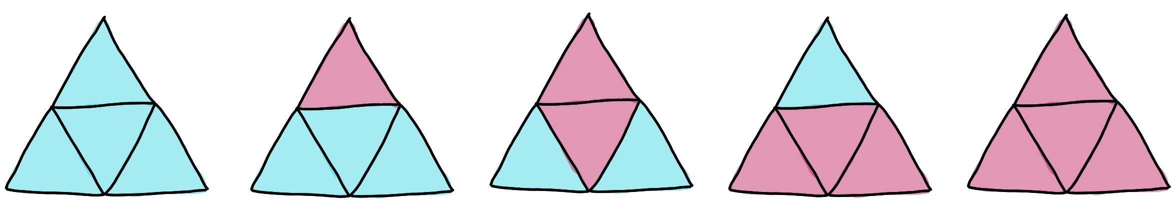
8 

k^2

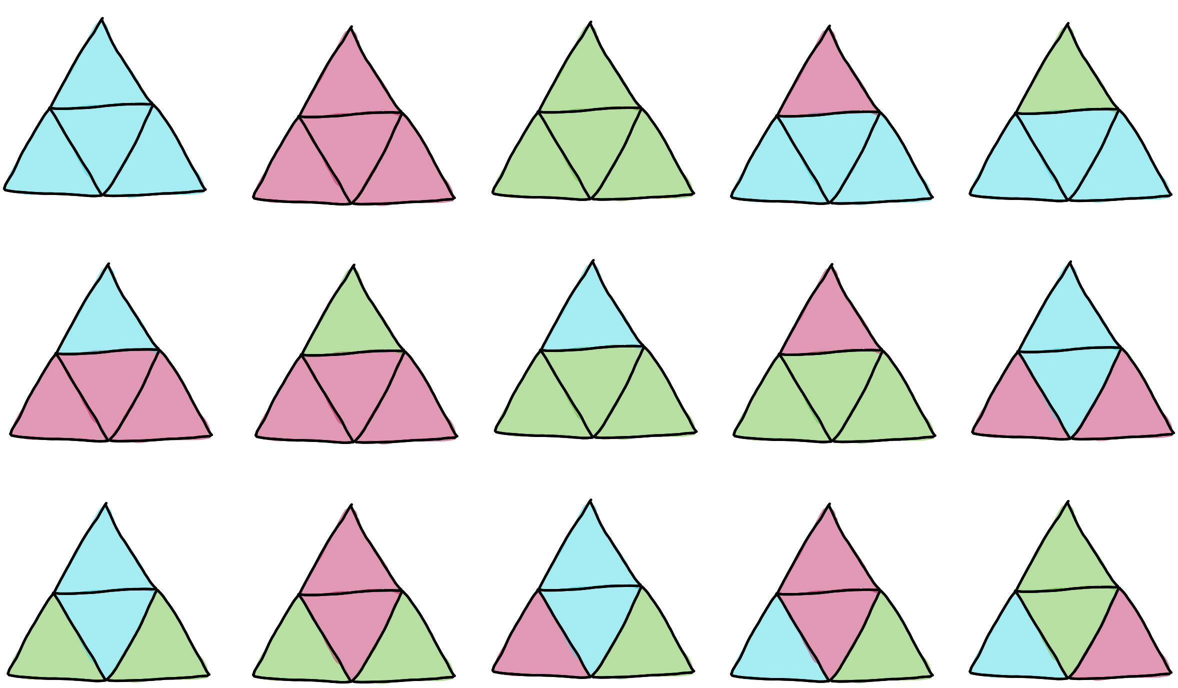
$$\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{12}\left(k^4 + 11k^2\right)$$

k	#	
1	1	
2	5	16+44
3	15	81+99
4	36	256+176

Check: k=2  5 ☑

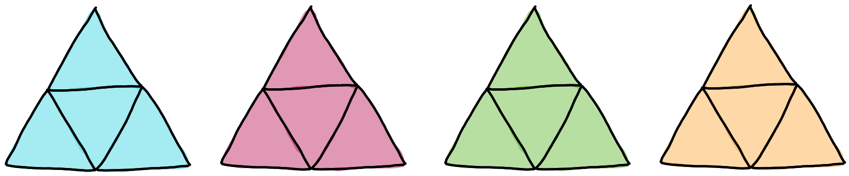


Check: k = 3  15 ☑

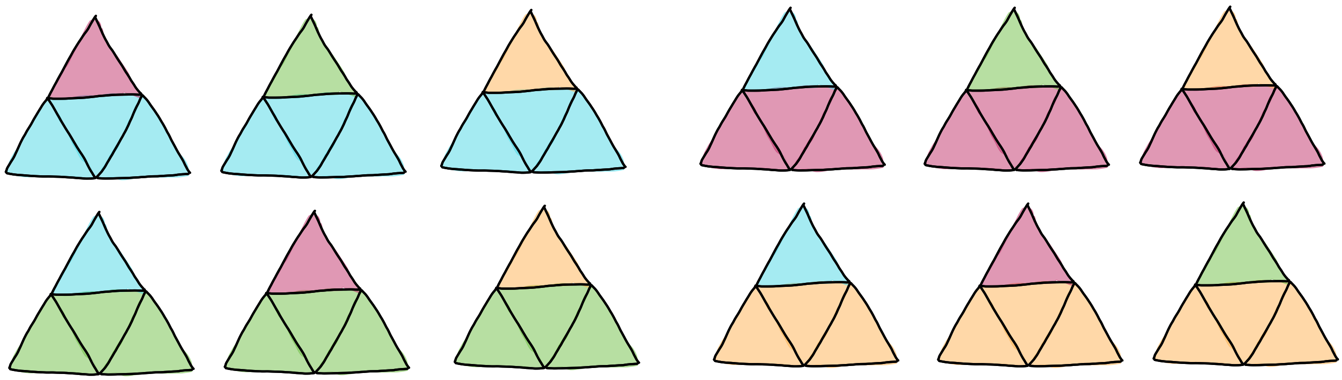


Check: k=4  36 ☑

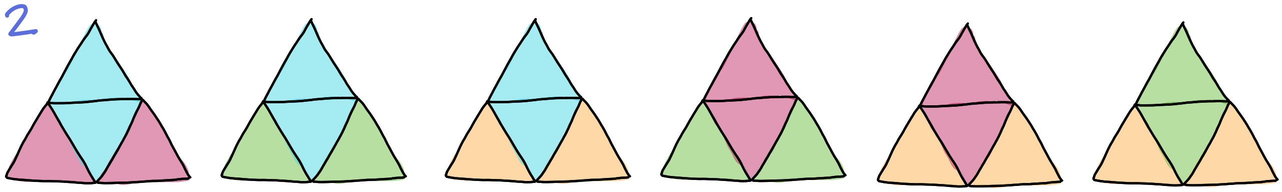
4



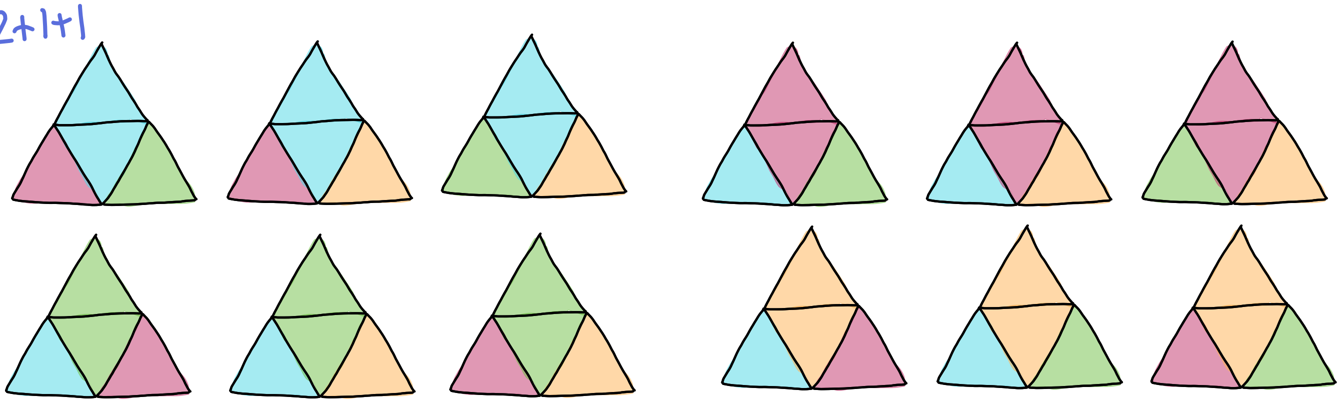
3+1

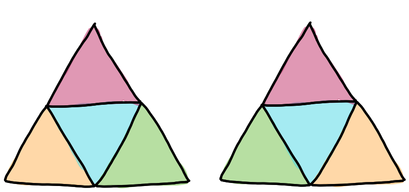


2+2



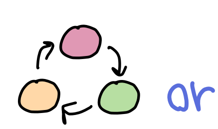
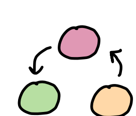
2+1+1





1+1+1+1

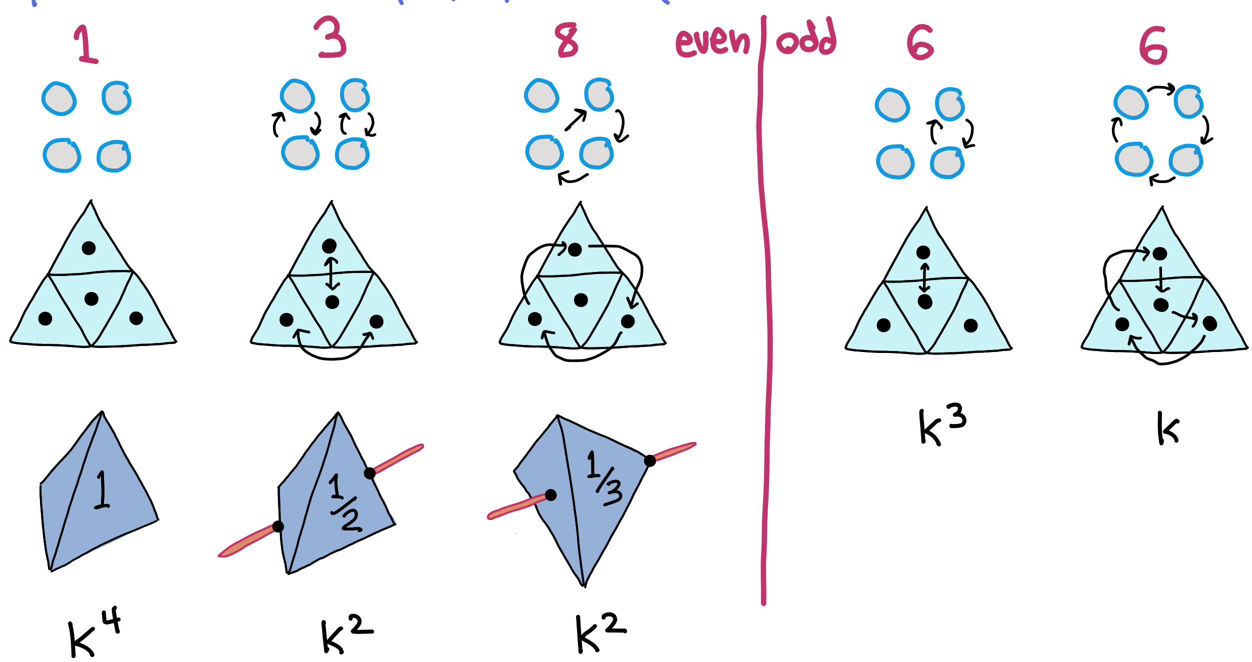
Finally a chiral pair

Look at , see  or 

This tells us that if we allow flips, we'll get

k	1	2	3	4	
G	1	5	15	36	(no flips)
S_4	1	5	15	35	(flips in $\mathbb{R}^4$)

$|S_4| = 4! = 24$ breaks up by cycle decomposition



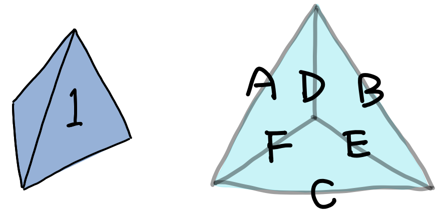
$$\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{24}\left(k^4 + 6k^3 + 11k^2 + 6k\right)$$

k	k^2	k^3	k^4	$6k$	$11k^2$	$6k^3$	k^4	Σ	#	
1	1	1	1	6	11	6	1	24	1	as before
2	4	8	16	12	44	48	16	120	5	as before
3	9	27	81	18	99	162	81	360	15	as before
4	16	64	256	24	176	384	256	840	35 ☑	not 36

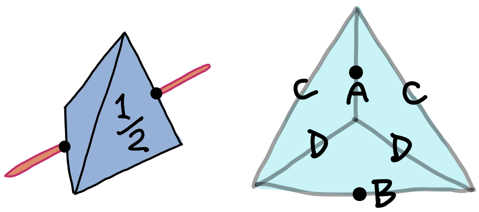
Choosing subsets of faces is restricted version of 2-coloring ⇒ no chirality
Coloring vertices is dual to coloring faces, same problem

- Coloring edges?
- Coloring everything?

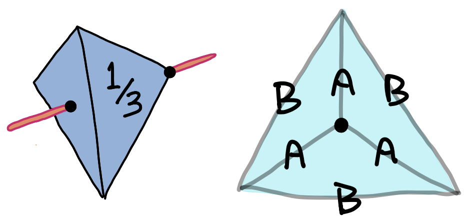
Coloring edges:

1 

k^6

3 

↔ k^4

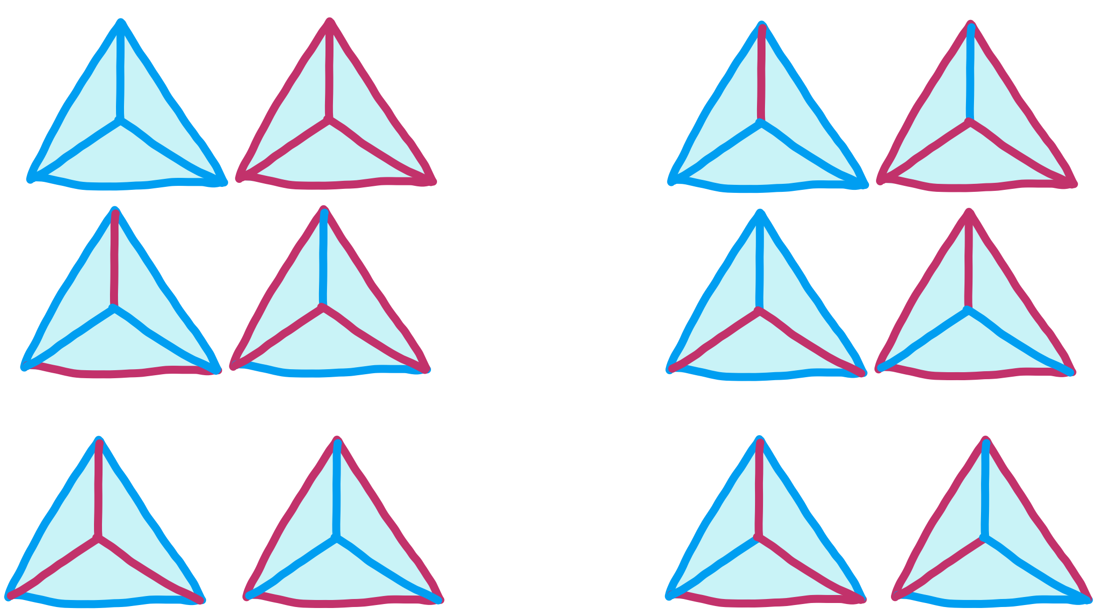
8 

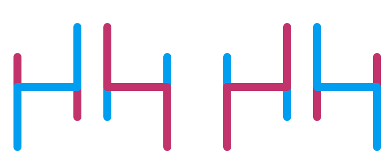
2 k^2

$$\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{12}\left(k^6 + 3k^4 + 8k^2\right)$$

k	1	2	3
k^2	1	4	9
k^4	1	16	81
k^6	1	64	729
$8k^2$	8	32	72
$3k^4$	3	48	243
k^6	1	64	729
Σ	12	144	1044
#	1	12	84

Check: k=2 12 ✓



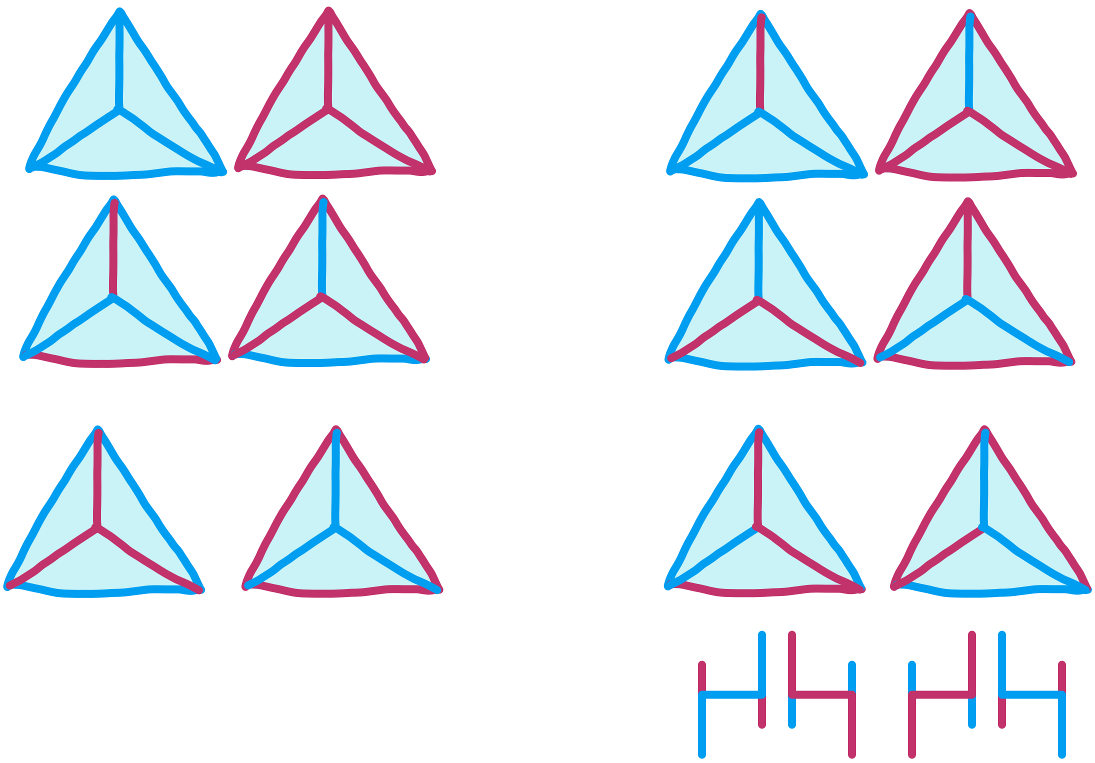


(Corrected from class)

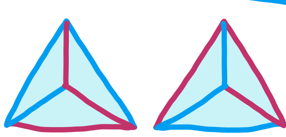
Tuesday, March 16

From last class: 12 ways to 2-color edges of a tetrahedron, up to symmetry

Check: k=2 12 ✓

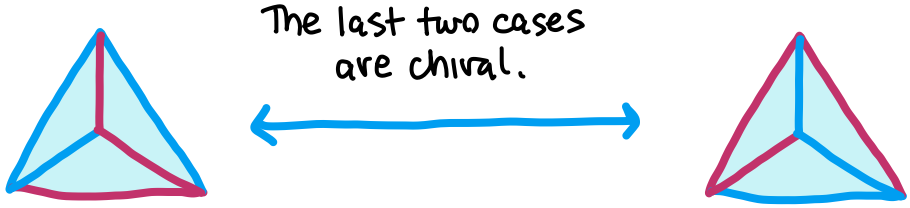


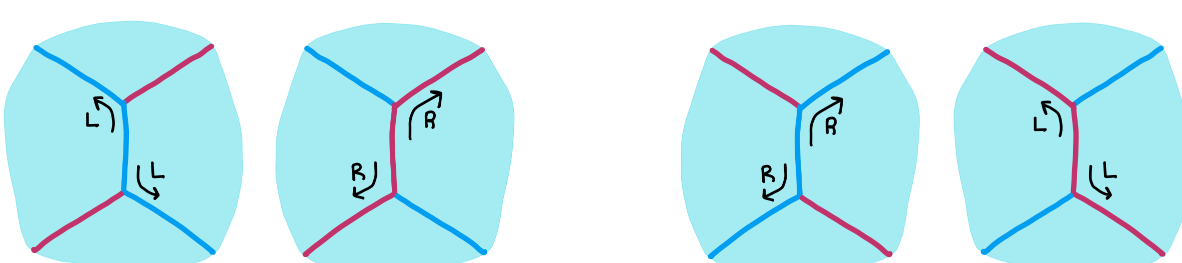
(Corrected from class)

In class I had: 

These were actually the same.

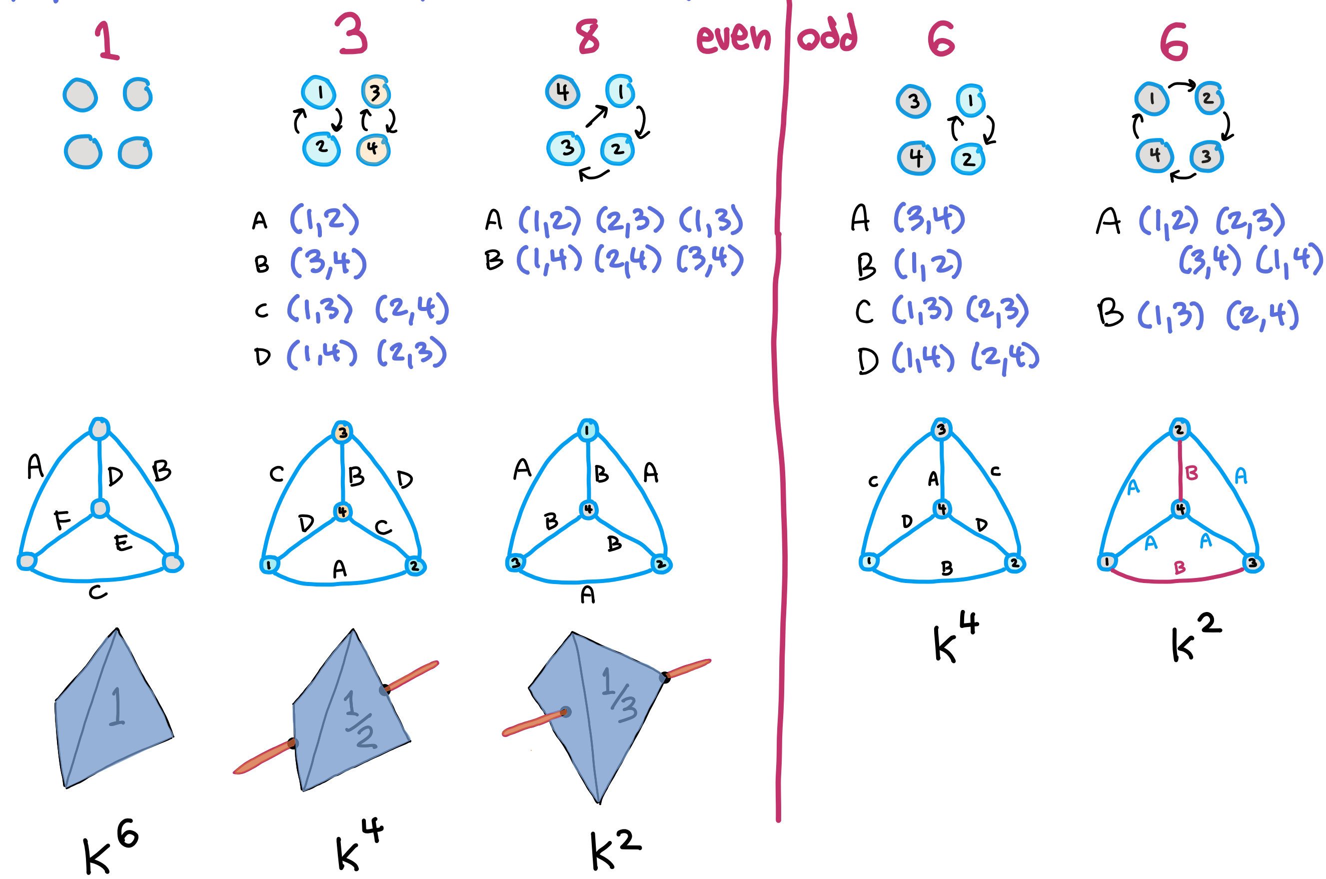
The last two cases are chiral.





This tells us that including flips through $\mathbb{R}^4$, we should get 11 not 12

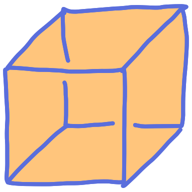
$|S_4| = 4! = 24$ breaks up by cycle decomposition



$$\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{24}\left(k^6 + 9k^4 + 14k^2\right)$$

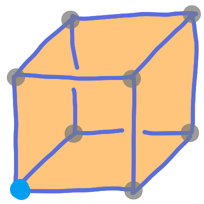
k	k^2	k^4	k^6	$14k^2$	$9k^4$	k^6	Σ	#
1	1	1	1	14	9	1	24	1
2	4	16	64	56	144	64	264	11 ☑

Symmetries of the cube

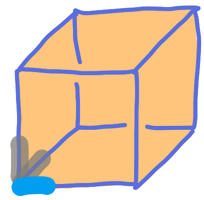


G = group of symmetries of cube in $\mathbb{R}^3$

(we ignore flips through $\mathbb{R}^4$)



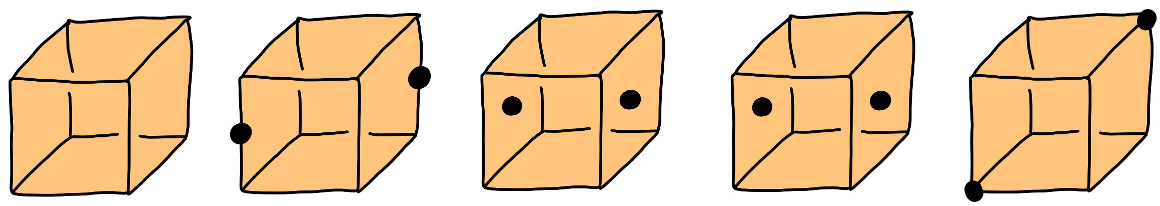
① Choose a corner
8 choices



② choose an edge meeting that corner
3 choices

$|G| = 8 \cdot 3 = 24$

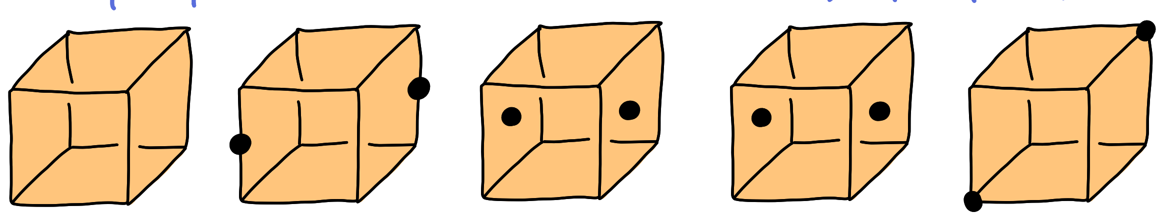
Can we find these 24 symmetries?



identity 1	½ turn	½ turn	¼ turn either way	⅓ turn either way
1	6	3	6	8

This $G \approx S_4$. Imagine 4 diagonal sticks inside the cube.
Easier: Label opposite corners the same, using $\{1,2,3,4\}$
Every permutation is possible.

How many ways can we k-color the faces of a cube, up to symmetry?



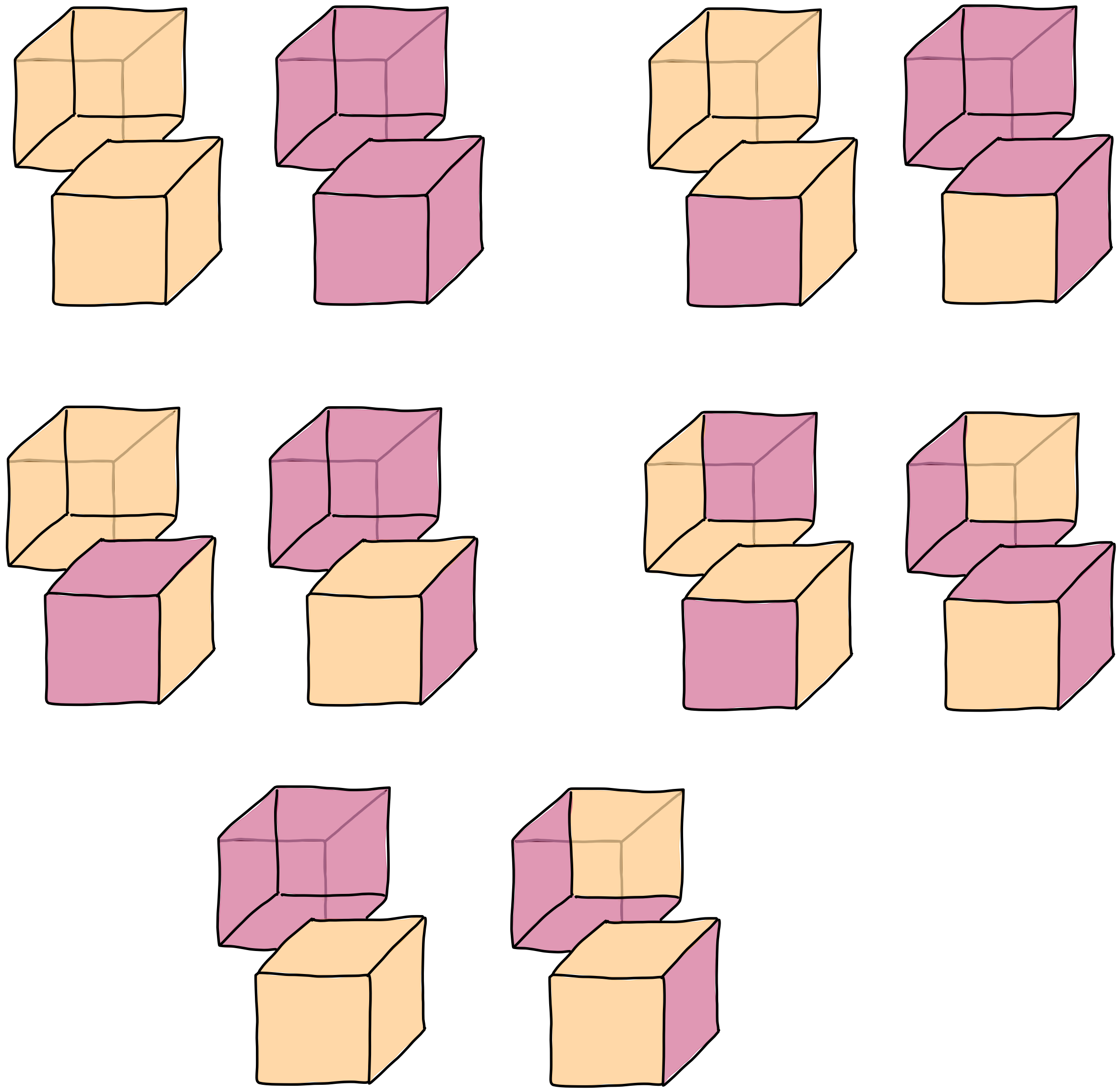
identity 1	½ turn	½ turn	¼ turn either way	⅓ turn either way
k^6	$6k^3$	$3k^4$	$6k^3$	$8k^2$

$\frac{1}{24}(k^6 + 3k^4 + 12k^3 + 8k^2)$

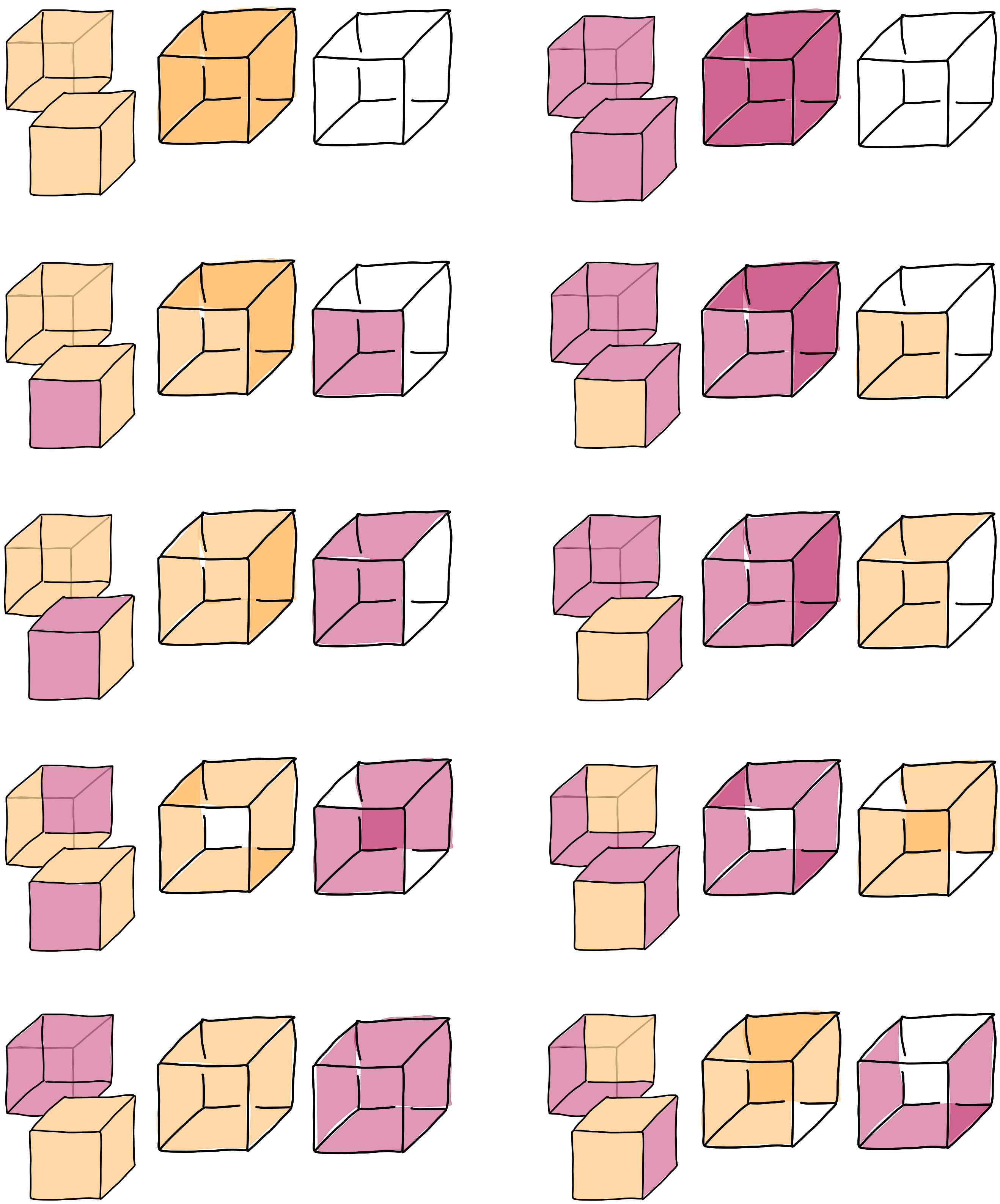
$k=2 \Rightarrow \frac{1}{24}(64 + 3\cdot16 + 12\cdot8 + 8\cdot4) = 240/24 = 10$

(48, 96, 32)

Check k = 2: 10



After class: Try another way to draw these.
Two wire frames per pattern, to separate the faces of each color.

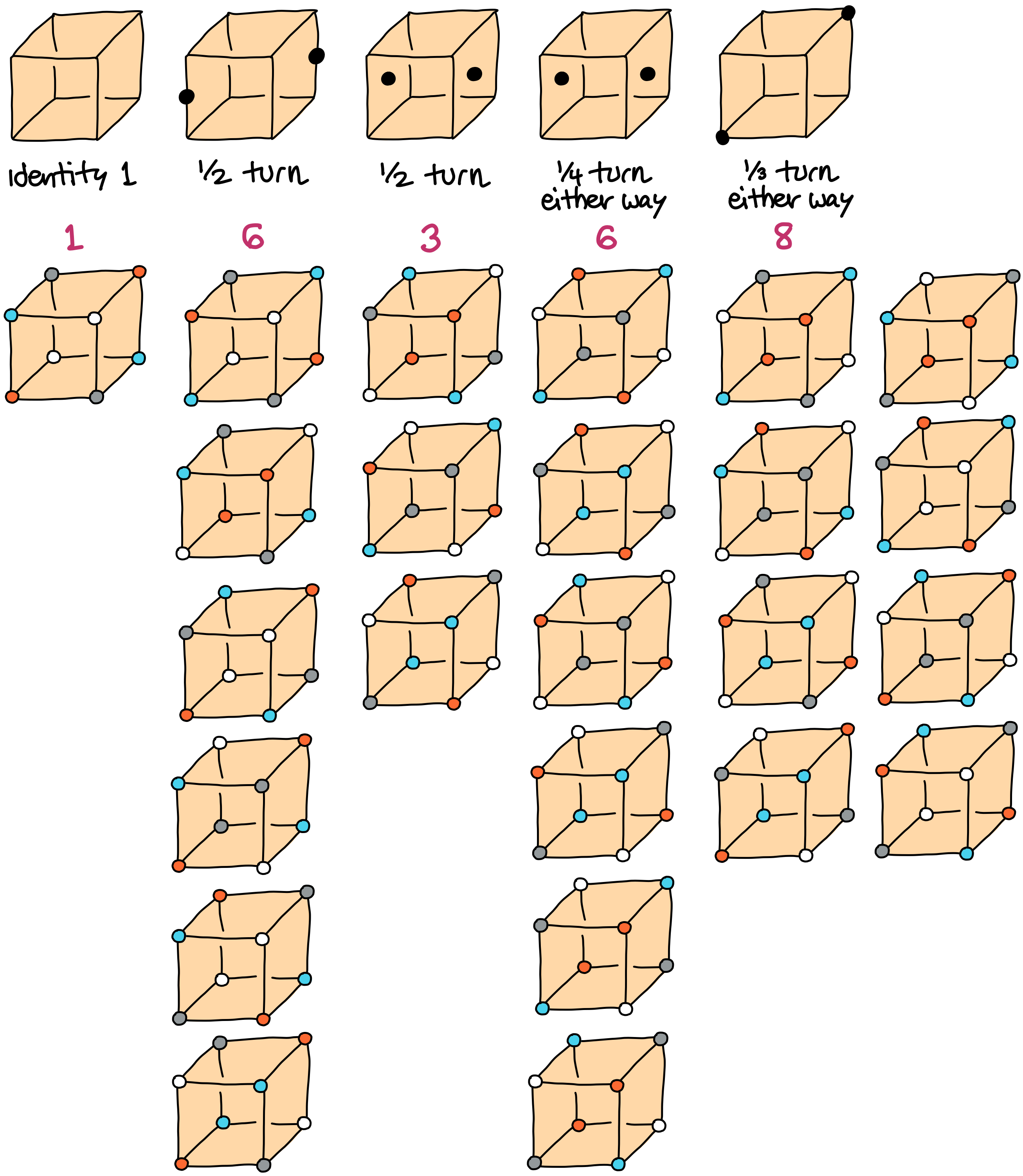


Check that action of S_4 induces every symmetry of cube

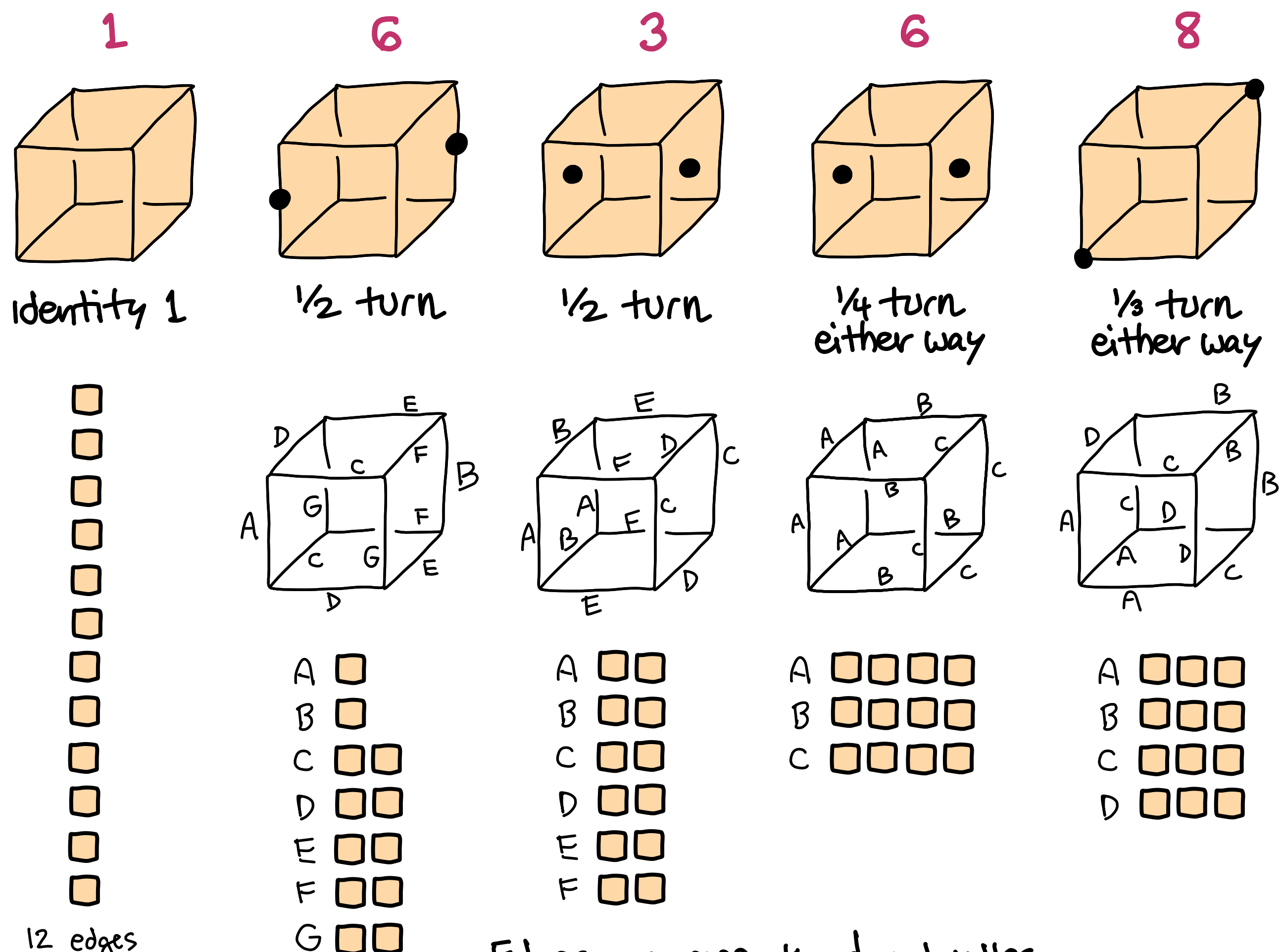
Four pairs of opposite corners, marked by

S_4 permutes these pairs

Every permutation corresponds to some rotation in space:



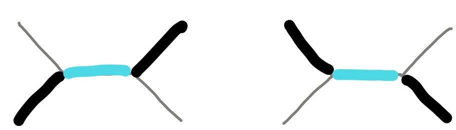
How many ways can we choose k edges of a cube, up to symmetry?



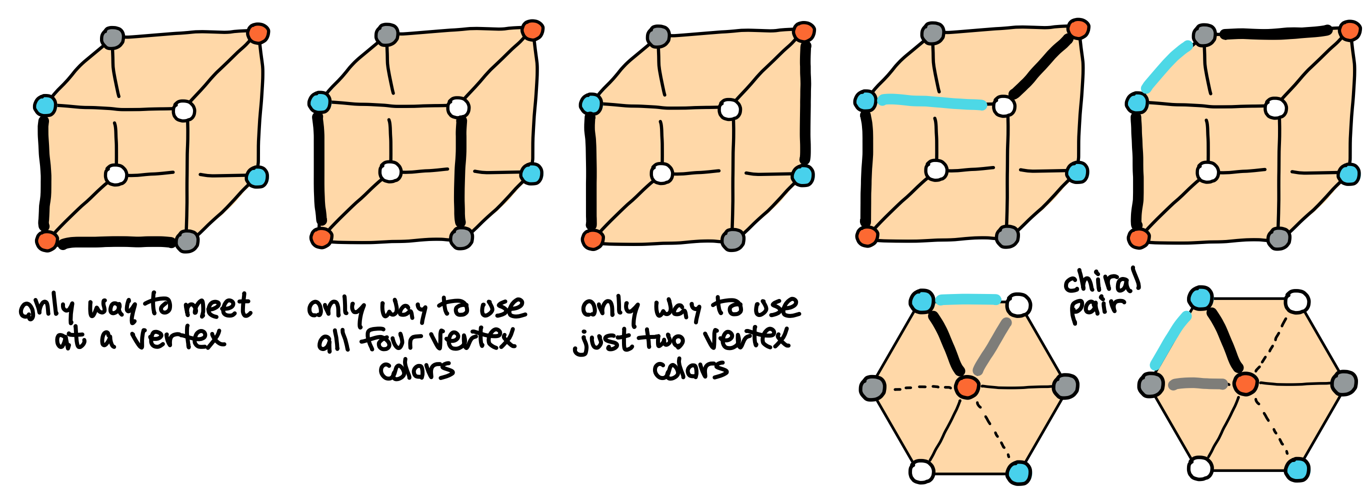
Edges come prepackaged in bundles
We need to make k buying entire bundles

K=2

$\binom{12}{2}=66$ 6 6 0 0



$$\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{24}(66+6\cdot 6+3\cdot 6) = \frac{120}{24} = 5$$

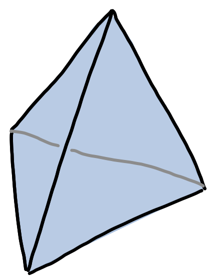


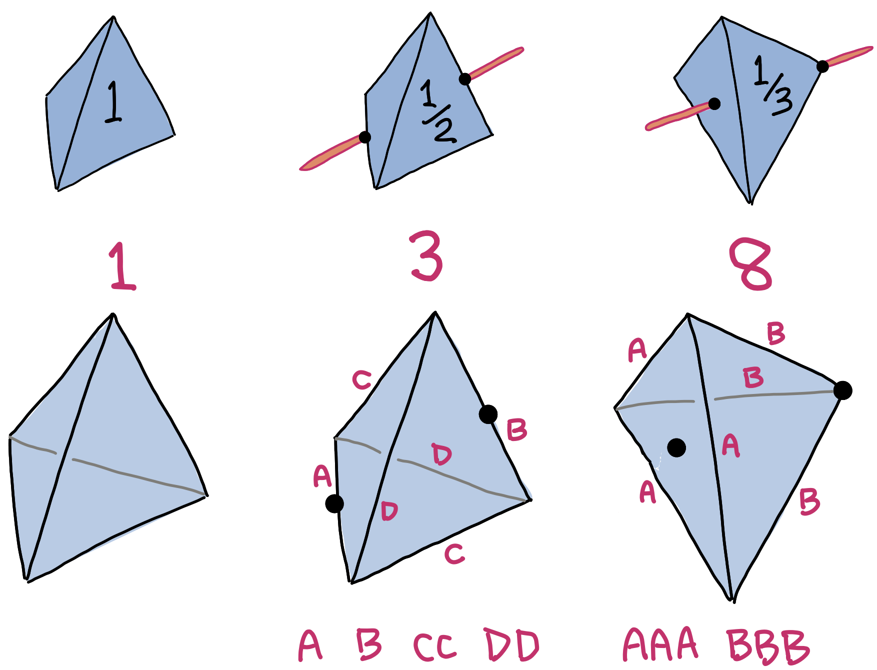
Exam 2

Combinatorics, Dave Bayer, March 18-21, 2021

To receive full credit for correct answers, please show all work.

[1] How many ways can we choose three edges of a regular tetrahedron, up to rotational symmetry? Confirm your answer by finding all patterns up to symmetry.

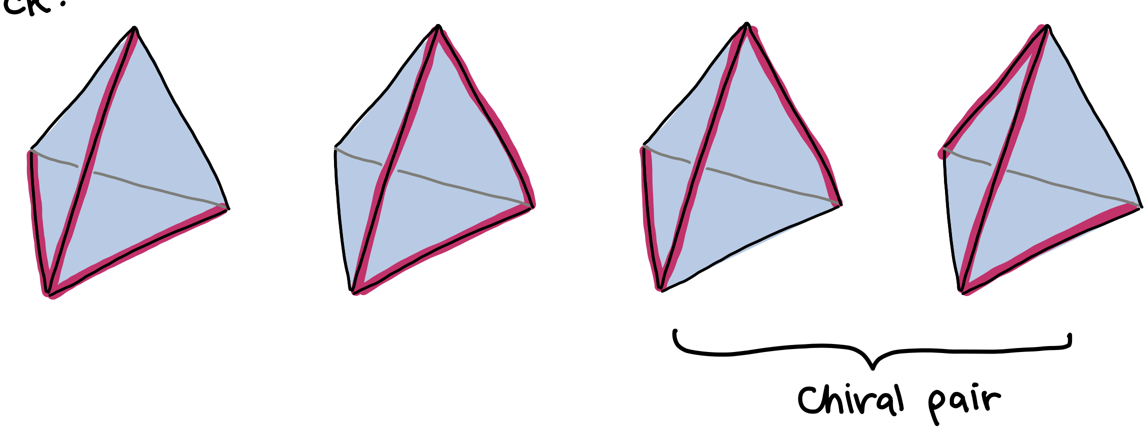




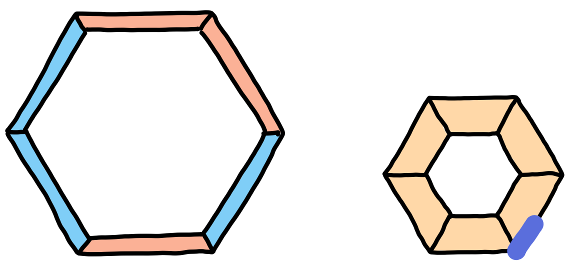
$\binom{6}{3}=20$ $\quad$ $2\cdot 2=4$ $\quad$ 2

$$\frac{1}{12}\left(20+\underbrace{3\cdot 4}_{12}+\underbrace{8\cdot 2}_{16}\right) = 48/12 = \boxed{4}$$

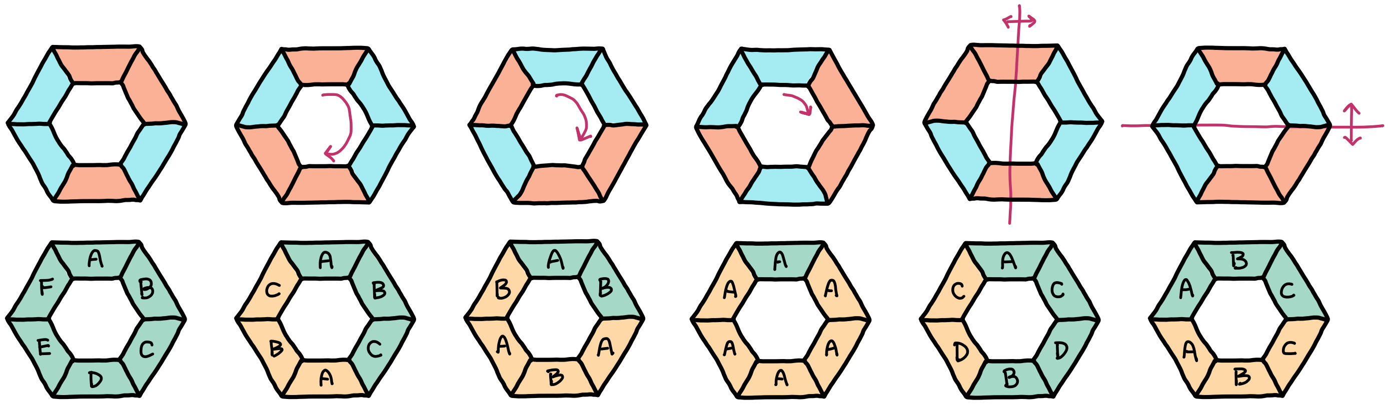
Check:



[2] How many ways can we k-color the six sides of a regular hexagon, up to rotational and flip symmetries? Confirm your answer for $k = 2$, by finding all patterns up to symmetry.



$|G| = 6$ vertices $\cdot$ 2 edges $= 12$ cases
6 rotations possibilities
6 flips

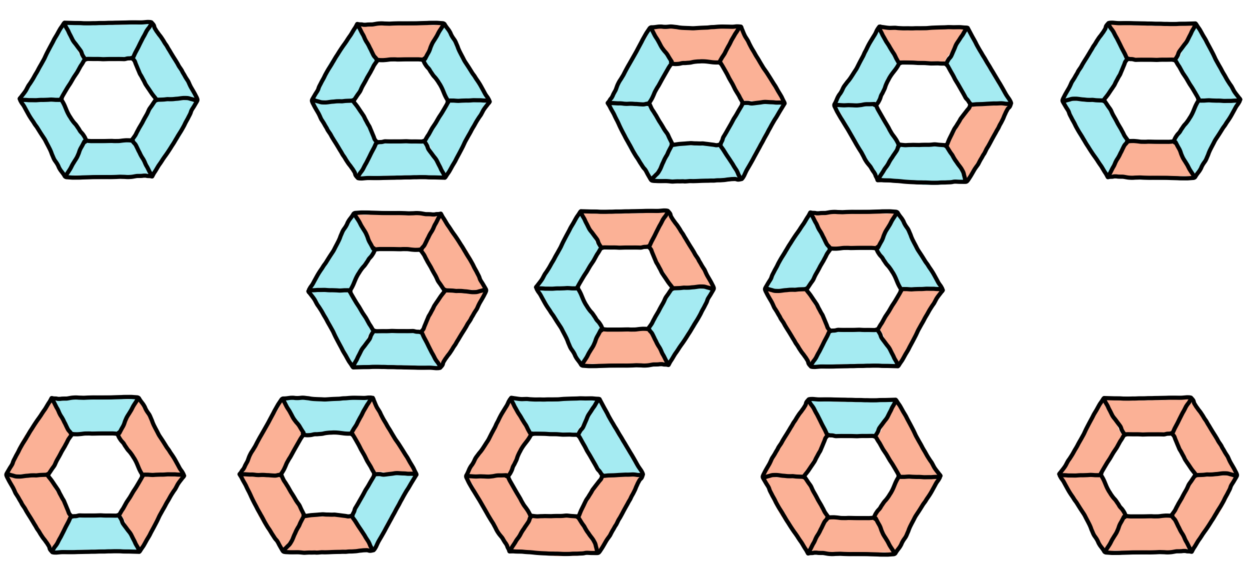


1	1	2	2	3	3
Identity	½ turn	⅓ turns	⅙ turns	side flips	vertex flips
k^6	k^3	k^2	k	k^4	k^3

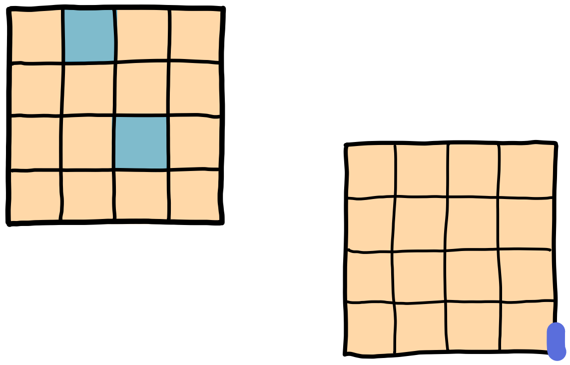
$$\boxed{\frac{1}{12}\left(k^6 + 3k^4 + 4k^3 + 2k^2 + 2k\right)}$$

$$k=2: \quad \frac{1}{12}\left(64 + \underset{48}{3\cdot 16} + \underset{32}{4\cdot 8} + \underset{8}{2\cdot 4} + \underset{4}{2\cdot 2}\right) = 156/12 = \boxed{13}$$

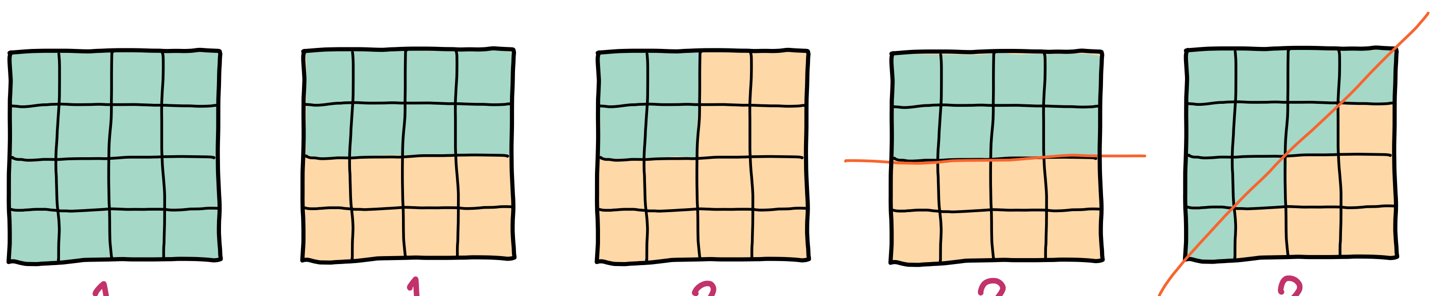
Check:



[3] How many ways can we choose two squares of a 4 × 4 board, up to rotational and flip symmetries? Confirm your answer by finding all patterns up to symmetry.



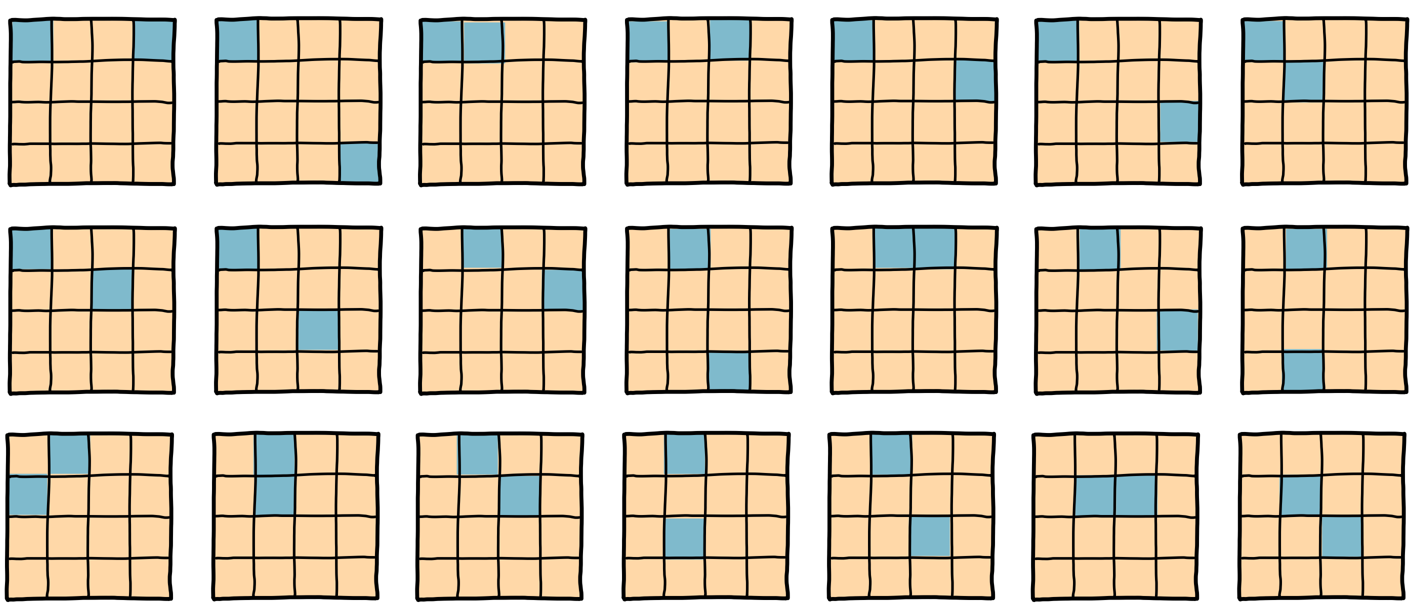
$|G| = 4 \text{ corners} \cdot 2 \text{ edges} = 8$



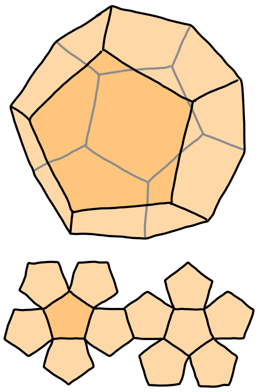
Identity	½ turn	¼ turns	side flips	vertex flips
1	1	2	2	2
$\binom{16}{2} = 8 \cdot 15 = 120$	8	0	8	$\binom{4}{2} + 6 = 12$

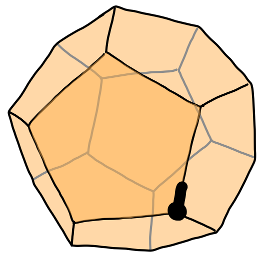
$$\frac{1}{8}\left(120 + 8 + \underset{16}{2 \cdot 8} + \underset{24}{2 \cdot 12}\right) = 168/8 = \boxed{21}$$

Check:

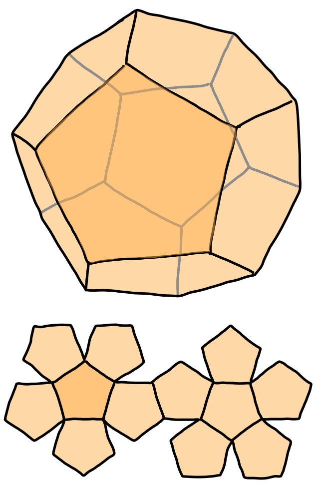


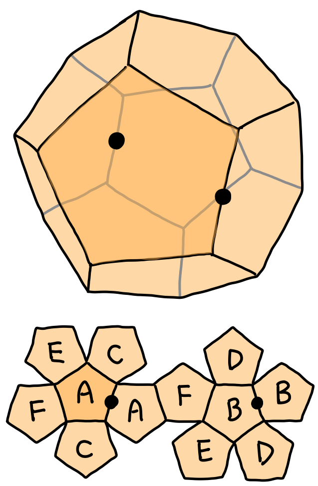
[4] How many ways can we choose 2 or 3 faces of a regular dodecahedron up to rotational symmetry? Confirm your answers by finding all patterns up to symmetry.

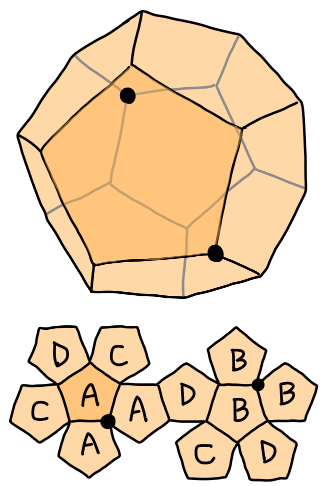


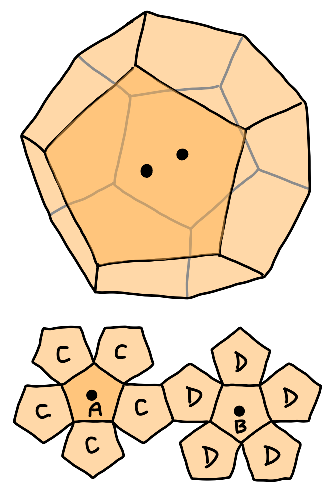


12 pentagon faces
30 edges $\quad 5\cdot 12/2$
20 vertices $\quad 5\cdot 12/3$
Choose vertex then edge
$|G| = 20\cdot 3 = 60$







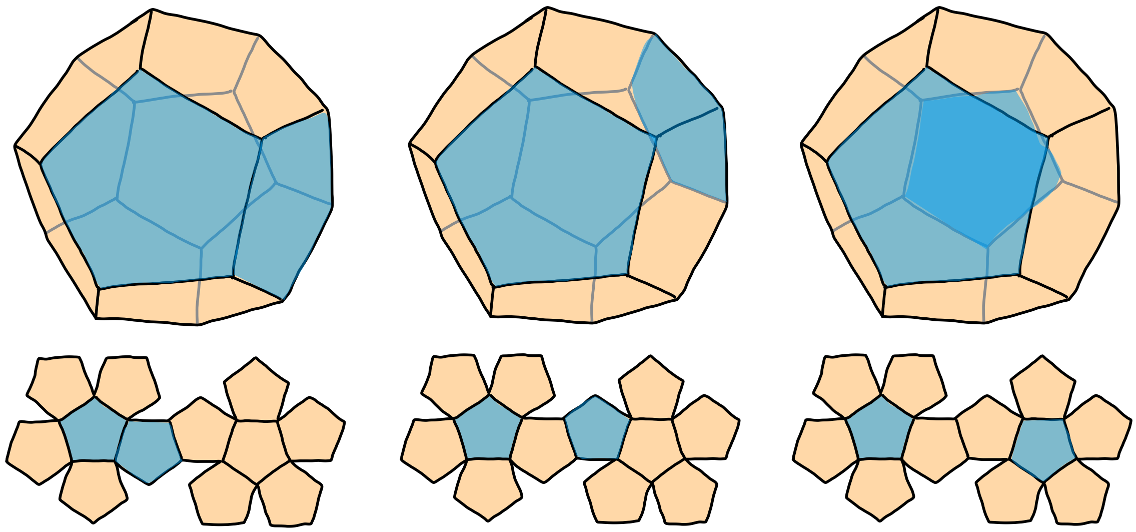


	1	15	20	24
	Identity	Edge ½ turns	Vertex ⅓ turns	Face turns
	$\frac{12\cdot 11}{2\cdot 1}$ $\quad \frac{12\cdot 11\cdot 10}{3\cdot 2\cdot 1}$	AA BB CC DD EE FF	AAA BBB CCC DDD	A CCCCC B DDDDD
k=2	$\binom{12}{2}=66$	6	0	1
k=3	$\binom{12}{3}=220$	0	4	0

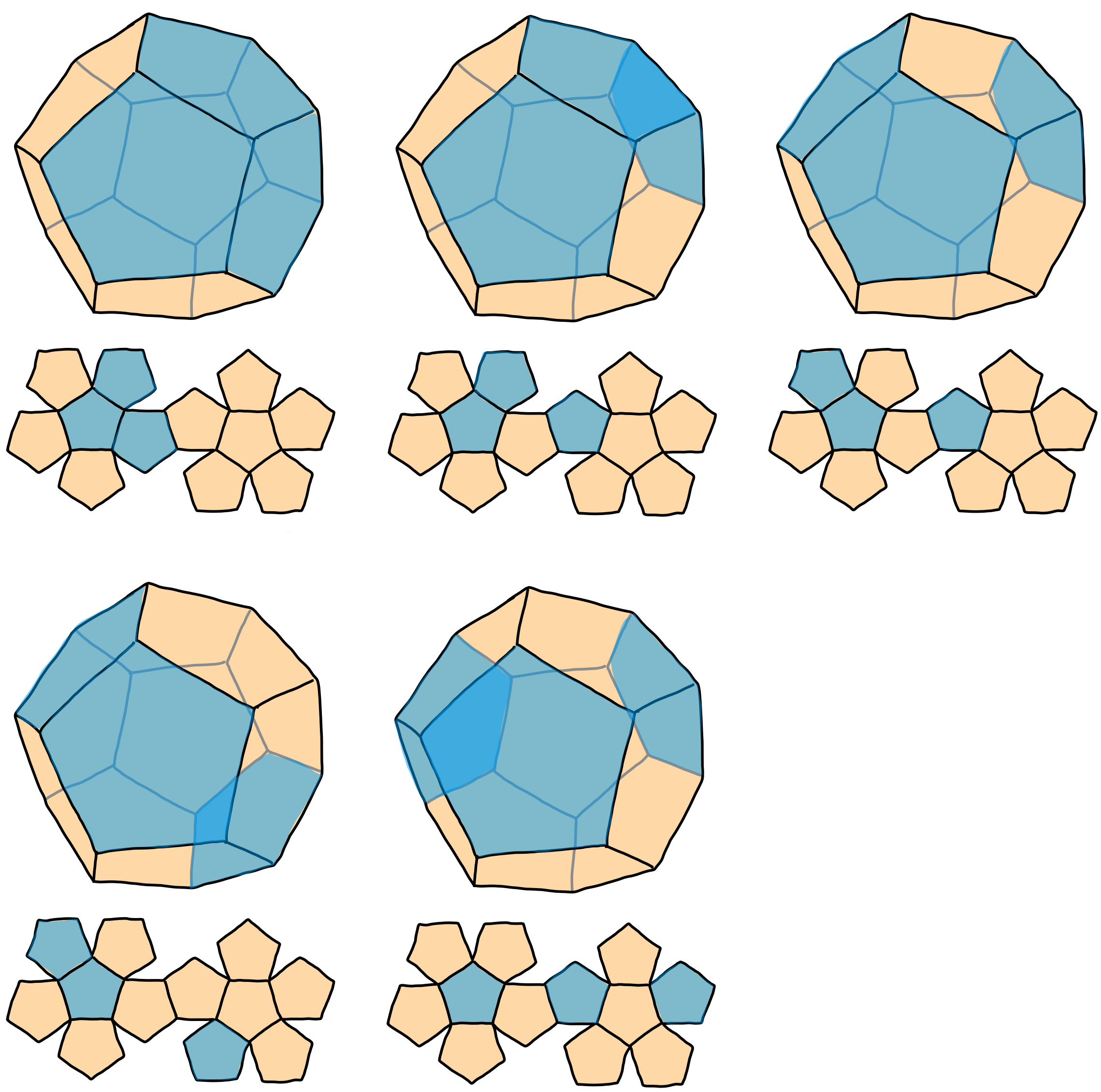
k=2 $\quad \frac{1}{60}(66 + 15\cdot 6 + 20\cdot 0 + 24\cdot 1) = 180/60 = \boxed{3}$

k=3 $\quad \frac{1}{60}(220 + 15\cdot 0 + 20\cdot 4 + 24\cdot 0) = 300/60 = \boxed{5}$

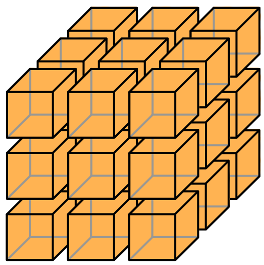
K=2 **3**

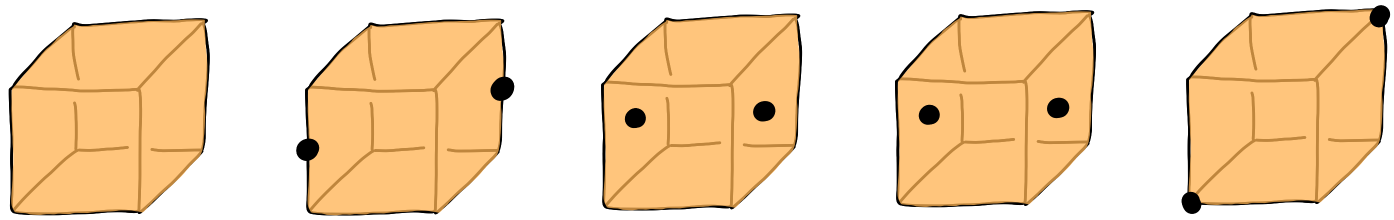


K=3 **5**

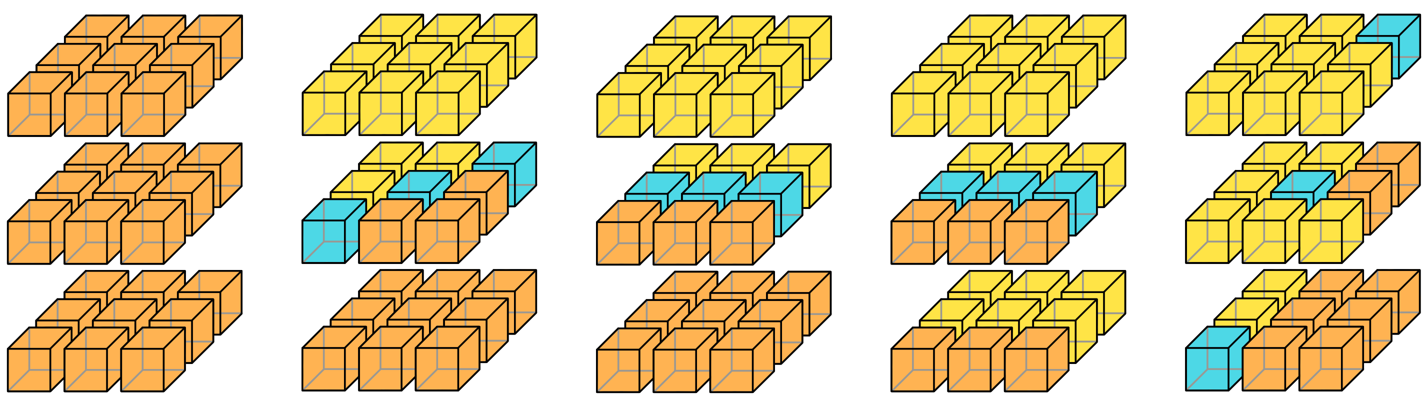


[5] How many ways can we choose two cubes from a $3 \times 3 \times 3$ array of 27 cubes, up to rotational symmetry? (This is not a *Rubik's Cube*. The symmetries are the 24 rotations we have studied of a solid cube.)





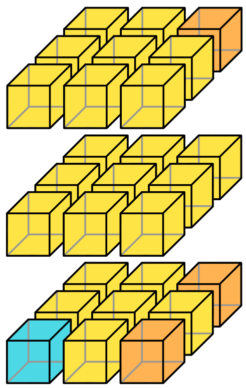
identity 1	½ turn	½ turn	¼ turn either way	⅓ turn either way
1	6	3	6	8



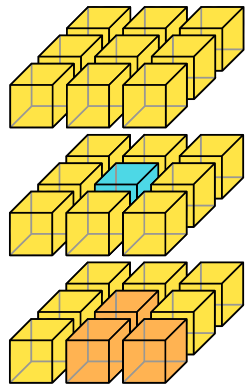
$\binom{27}{2} = \frac{27 \cdot 26}{2 \cdot 1} = 351$	3 cubes on axis 12 pairs $\binom{3}{2} + 12 = 15$	3 cubes on axis 12 pairs $\binom{3}{2} + 12 = 15$	3 cubes on axis 6 quads $\binom{3}{2} = 3$	3 cubes on axis 8 triplets $\binom{3}{2} = 3$

$\frac{1}{24}(351 + 9 \cdot 15 + 14 \cdot 3) = 528/24 = \boxed{22}$ ways to pick two cubes

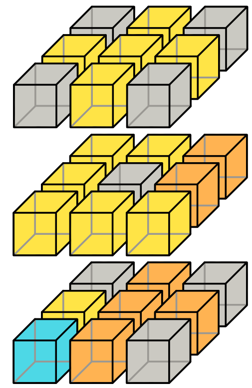
Check:



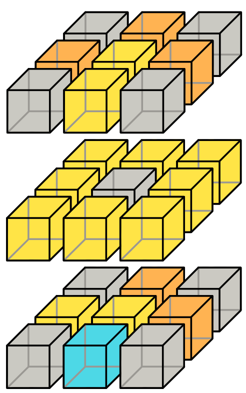
3 ways to choose two corners



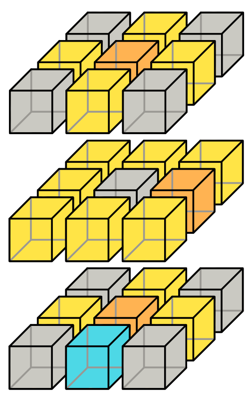
3 ways to choose middle



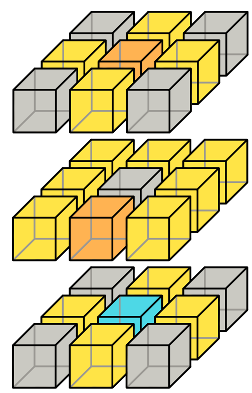
6 ways left to choose one corner



5 ways to choose two edges (as we saw before)



3 ways to choose one edge, one face



2 ways to choose two faces

$$3+7+2+5+3+2 = \boxed{22}$$ ☑