INTRODUCTION TO 6 FUNCTOR FORMALISMS

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A six functor formalism naturally arises in various sheaf theories. Roughly speaking, it consists of a collection of 6 functors on the derived category of sheaves on a space or spaces which can be used to encode their cohomology. Isomorphisms between certain compositions of them can express deep theorems, the central one being the Poincaré duality theorem[1]. Six functor formalisms were originally developed in the context of coherent cohomology, étale cohomology and cohomology of ordinary topological spaces by Grothendieck and his school. However, its range of applicability is impressive: coherent cohomology, Betti cohomology, l-adic cohomology, D-modules, mixed Hodge modules, etc. Grothendieck once explained that discovering how this formalism held in both extreme cases of continuous coefficients (in coherent cohomology) and discrete coefficients (in étale cohomology) convinced him of their ubiquity in geometric situations giving rise to duality theorems.

1. Étale cohomology

While we will work in the context of étale cohomology, all the essential features described hold in other scenarios. For a similar discussion for (nice) topological spaces, see [Sch22], Lecture 1.

1.1. **Definition of the 6 functors.** Let X be a scheme. Given an étale sheaf $\mathcal{F}$ on X, we can thus encapsulate all the cohomology groups $H^i_{\text{ét}}(X, \mathcal{F})$ with the single derived functor $R\Gamma(X, -)\colon D(X_{\text{ét}}) \to D(\mathbf{Ab})$. More generally, given a morphism of schemes $f\colon X \to Y$, we can construct the derived functor $Rf_*\colon D(X_{\text{ét}}) \to D(Y_{\text{ét}})$ as follows. We define

$$f_*\mathcal{F}(U) := \mathcal{F}(U \times_Y X)$$

as an étale sheaf on Y. Then Rf_* is the derived functor of f_*.

Warning: When working with derived categories, it is common to drop the 'R' in front of Rf_* and just use f_* to denote Rf_*, because everything is assumed to be derived. We will follow this convention from now on – everything is derived.

Now f_* has a left adjoint: $f^*\colon D(Y_{\text{ét}}) \to D(X_{\text{ét}})$. One may define it on the sheaf level as the sheafification of the presheaf

$$P(U) := \varinjlim \mathcal{F}(V)$$

over commutative diagrams

$$\begin{array}{ccc} U & \longrightarrow & V \\ {\scriptstyle etale}\downarrow & & \downarrow{\scriptstyle etale} \\ X & \longrightarrow & Y \end{array} .$$

with structure morphism $f : X \to \operatorname{Spec} k$. its cohomology groups can be written as $\Gamma(\operatorname{Spec} k, R^i f_*\mathcal{F}) \cong H^i(X, \mathcal{F})$.

In the case $Y = \operatorname{Spec} k$ is a point, we can express the cohomology of X with constant coefficients very concisely with these functors as $f_*f^*\mathbb{Z}/l$. More generally, we can view f_* as a

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[1]To be clear, Poincaré duality here refers to the duality theorem in each instantiation of a 6 functor formalism, the classical Poincaré duality being the one for manifolds.

relative version of cohomology, in the sense that $Rf_*\mathcal{F}$ restricted to the points $y \in Y$ are related to the cohomology of the fibers X_y. More precisely, for every cartesian square below, there is a base change map $g^*f_* \to f'_*g'^*$, which however is not always an isomorphism.

$$\begin{array}{ccc} U & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow[g]{} & Y \end{array}$$

It is an isomorphism if $\mathcal{F}$ is torsion and f is proper, so this motivates the definition of $f_!$. By the Nagata compactification theorem, if f is separated and of finite type, there is a factorization $f = p \circ j$ where j is an open immersion and p is proper.

$$\begin{array}{ccc} X & \xrightarrow{j} & \overline{X} \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} \\ & & Y \end{array}$$

Then we define

$$f_! := p_*j_!,$$

where $j_!$ is extension by 0. The functor $f_!$ is known as proper pushforward, and if f is an open immersion then f^* is its right adjoint. In general $f_!$ does possess a right adjoint known as $f^!$, but its existence is quite non-trivial. However, if f is smooth of relative dimension d, then $f^! = f^*[2d](d)$.

Finally, the last two functors are given by derived tensor product and derived sheaf hom, and they are adjoint as well:

$$\mathrm{Hom}(A, \underline{\mathrm{Hom}}(B, C)) \cong \mathrm{Hom}(A \otimes B, C).$$

1.2. **Main properties.** Together, these functors allow us to express several powerful results, which we will now state. The standing assumptions are that $f\colon X \to Y$ is a morphism of schemes, $\mathcal{F} \in D(X_{\text{ét}}), \mathcal{G} \in D(Y_{\text{ét}})$ are torsion sheaves.

(1) (Proper base change) In the cartesian square below, the natural base change morphism

$$g^*f_!\mathcal{F} \to f'_!g'^*\mathcal{F}$$

is an isomorphism.

$$\begin{array}{ccc} U & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow[g]{} & Y \end{array}$$

(2) (Projection formula) The natural morphism

$$f_!\mathcal{F} \otimes \mathcal{G} \to f_!(\mathcal{F} \otimes f^*\mathcal{G})$$

is an isomorphism. If we replace $f_!$ with f_*, such a morphism always exists by adjunction. Indeed, it is adjoint to the map

$$f^*(f_*\mathcal{F} \otimes \mathcal{G}) \cong f^*f_*\mathcal{F} \otimes f^*\mathcal{G} \to \mathcal{F} \otimes f^*\mathcal{G}.$$

(3) (Künneth formula) There is a natural isomorphism

$$R\Gamma_c(X, \mathcal{F}) \otimes R\Gamma_c(Y, \mathcal{G}) \cong R\Gamma_c(X \times Y, \mathcal{F} \boxtimes \mathcal{G}).$$

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_1} & X \\ {\scriptstyle p_2}\downarrow & \overset{p}{\searrow} & \downarrow{\scriptstyle p_X} \\ Y & \xrightarrow[p_Y]{} & * \end{array}$$

This is a formal consequence of the previous two results.

Proof. We have

$$p_!(\mathcal{F} \boxtimes \mathcal{G}) = p_{X!}p_{1!}(p_1^*\mathcal{F} \otimes p_2^*\mathcal{G}) \cong p_{X!}(\mathcal{F} \otimes p_{1!}p_2^*\mathcal{G}) \cong p_{X!}(\mathcal{F} \otimes p_X^* p_{Y!}\mathcal{G}) \cong p_{X!}\mathcal{F} \otimes p_{Y!}\mathcal{G}.$$

□

(4) (Poincaré duality) The functors $(f_!, f^!)$ form an adjoint pair, and we have a natural isomorphism

$$f_* \underline{\mathrm{Hom}}_X(\mathcal{F}, f^!\mathcal{G}) \xrightarrow{\cong} \underline{\mathrm{Hom}}_Y(f_!\mathcal{F}, \mathcal{G}).$$

Applying this to $f : X \to \operatorname{Spec} k$ where X is smooth of dimension d and k is algebraically closed, we recall that in this case $f^!\mathbb{Z}/l \cong \mathbb{Z}/l[2d](d)$, and (modulo some details) we obtain the classical statement that $H^i(X, \mathcal{F}^\vee) \cong H_c^{2d-i}(X, \mathcal{F})^\vee$.

2. Modern formulation

The six functors formalism has not only been developed over different cohomology theories, but also over many different kinds of spaces, from topological spaces to schemes to stacks to rigid analytic spaces. In [Sch22], Scholze presents some unifying guidelines using ∞-categories that gives a very general framework for working with them. Here are some of the features of this approach.

- Given X a geometric object, $D(X)$ is treated as a stable ∞-category.
- Proving the duality theorem in different situations is generally difficult and heavily uses aspects specific to the scenario. However, some important common steps are identified and accomplished in this general framework.
- The six functors constructed satisfy various isomorphisms (mentioned in the previous section for étale cohomology), along with compatibility relations between these isomorphisms.
- These compatibilities are encoded as coherences, which are dealt with by ∞-categories.
- In particular, Mann presents the following approach in his thesis [Man22]. One begins with a category (ordinary, or ∞-) C and an appropriate class of morphisms E, and constructs a symmetric monoidal ∞-category (C, E) of correspondences.
- One defines a 3-functor formalism as a lax symmetric monoidal functor

  $$D \colon (C, E) \to \infty - \mathbf{Cat}.$$

  The three functors $\otimes, f^*, f_!$ and all their relations are naturally defined by the structure of the category.
- A 6-functor formalism is a 3-functor formalism where the three functors admit right adjoints. That's all!

References

[Man22] Lucas Mann. A p-adic 6-functor formalism in rigid-analytic geometry. page https://arxiv.org/abs/2206.02022, 2022.

[Sch22] Peter Scholze. Six-functor formalisms. https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf, October 2022.