1 Simplicial sets and ∞-categories

1.0.1. Define the simplicial category Δ consisting of finite sets Δ^n as objects and non-decreasing maps. Then define sSet as the category of pre-sheaves on simplicial sets; alternatively, it is the Ind-completion of Δ via Yoneda-embedding. Stress the importance of where the maps go for composition.

1.0.2. Given a simplicial set C, we define $C_n := C(\Delta^n)$, which may be described as $\mathrm{Mor}_{\mathrm{sSet}}(\Delta^n, C)$. For any category $\mathcal{C}$, we may define it's nerve $N(\mathcal{C})$, which has $\mathcal{C}_0$ as the set of objects, $\mathcal{C}_1$ as the set of morphisms, and $\mathcal{C}_2$ as the set of comutative triangles. The association $\mathcal{C} \longmapsto N(\mathcal{C})$ is a fully faithful embedding of the strict 1-category of categories to simplicial sets.

1.0.3. A morphism of simplicial sets $f : C \longrightarrow D$ is a *Kan fibration* if there is an extension:

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & C \\ \big\downarrow & \nearrow & \big\downarrow \\ \Delta^n & \longrightarrow & D \end{array}$$

An ∞-category is a simplicial set C for $\mathrm{Map}(\Delta^2, C) \longrightarrow \mathrm{Map}(\Lambda^2_1, C)$ is a Kan fibration. Here, Map is the same as Hom_{sSet}, the enriched hom in the category of simplicial sets, explicitely given by maps $\mathrm{Map}(C, D)_n := \mathrm{Hom}_{sSet}(C \times \Delta^n, D)$. It turns out that an equivalent condition for C to be an infinity category is that any map $\Lambda^n_i \longrightarrow C$ extends to map from the full n-simplex Δ^n for $0 < i < n$. We note $A \longrightarrow \mathrm{Map}(\Delta_0, A)$ is an isomorphism, and there is a natural evaluation morphism $A \times \mathrm{Map}(A, D) \longrightarrow D$.

1.0.4. There is a Quillen adjunction $\mathrm{sSet} \underset{S(-)}{\overset{|-|}{\rightleftarrows}} \mathrm{HoTop}$. This is not an equivalence, for example, the essential image of $S(-)$ are Kan complexes, which satisfy the filling horn condition for all $0 \leq i \leq n$. Other examples of Kan complexes include any *simplicial group*, and the ∞-categories for which are Kan complexes are called ∞-groupoids. The point of all this is if one is only interested in homotopy theory, one could just work with simplicial sets.

1.0.5. (i) For any ∞-category, and any simplicial set D, $\mathrm{Map}(D, C)$ is an infinity category. These form the *functor* category for an infinity category, and so we see infinity categories are enriched over themselves.

(ii) For an infinity category C, one can define its homotopy category, which consits of the objects of C and morphisms to be homotopy classes of maps (maps $f, g : X \longrightarrow Y$ for which there is a 2-simplex witnessing that $g : X \longrightarrow Y$ is the composite of $f : X \longrightarrow Y$ and id_Y.) Check this is a category.

(iii) $f : X \longrightarrow Y$ is defined to be an isomorphism if it is an isomorphism in $\mathrm{Ho}(C)$ (check this is an equivalence relation).

(iv) An ∞-groupoid is an ∞-category C such that all morphisms is invertible, equivaletly that $\mathrm{Ho}(C)$ is a groupoid. (Check equivalent to Kan complex).

(v) If X, Y are Kan complexes, then $\mathrm{Map}(X, Y)$ is also a Kan complex. This makes ∞-groupoids enriched over themselves.

Note: With objects infinity categories, and the infinity category of functors as morphisms, we use the homotopy coherent nerve functor to get the infinity category of infinity categories.

1.0.6. We will eventually get to the notion of a symmetric monoidal ∞-category. For now, let us note that $(C, \otimes)$, and X an object in C, we should have unit map $1 \longrightarrow X$ and multiplication map $X \otimes X \longrightarrow X$ satisfying homotopy coherent associativty constraints. Furthermore, a functor $F : (C\otimes) \longrightarrow (D, \otimes)$ between symmetric monoidal categories, should be a lax symmetric monoidal functor, which means it has maps $1_D \longrightarrow F(1_C)$ and $F(X) \otimes F(Y) \longrightarrow F(X \otimes Y)$ functorial in $X, Y \in C$ with again higher homotopical coherence data.

1.0.7. The symmetric monoidal ∞-category of correspondences $\mathrm{Corr}(C, E)$ consists of an ∞-category C admitting finite limits, and a class of morphisms E stable under pullbacks and compositions containing all isomorphisms, and the symetric monoidal structureis given as follows:

(i) The objects are the objects of C

(ii) The symmetric monoidal structure is the Cartesian symmetric monoidal structure on C.

(iii) $\mathrm{Hom}_{\mathrm{Corr}(C,E)}(X, Y)$ is given by the ∞-groupoid of objects $W \in C$ together with maps $X \leftarrow W \rightarrow Y$ where $W \longrightarrow Y$ is in E.

(iv) Using fibre products one has composition.

1.0.8. A 3-functor formalism is a lax symmetric monoidal functor $D : \mathrm{Corr}(C, E) \longrightarrow \mathrm{Cat}_\infty$, where the right hand side is equipped with the product of infinity categories $\times$ structure. We see

(i) We have an association, for every object $X \in C$, an infinity category $D(X)$.

(ii) For any map $f : X \longrightarrow Y$, which defines a correspondence $Y \leftarrow X \rightarrow X$, we get a map $f^* : D(Y) \longrightarrow D(X)$¿

(iii) We have maps $D(X) \otimes D(X) \longrightarrow D(X \times X) \longrightarrow D(X)$, the right hand side from previous property applied to diagonal map, so the essential image of D admits a tensor product.

(iv) For $f : X \longrightarrow Y$ in E, we get a correpsondance $X \leftarrow X \rightarrow Y$ inducting a map $f_! : D(X) \longrightarrow D(Y)$.

A six functor formalism occurs when $\otimes, f^*, f_!$ admits right adjoints.