0.1. We set the following conventions:

- Cat_Δ to denote the sub-category of Cat consisting of simplicially enriched categories.
- s$\mathcal{C}$ to be the category of simplicial objects in a category $\mathcal{C}$.

1 Homotopy coherent nerve and correspondences

1.1. **Homotopy coherent nerve:** *See here for more details.*
There is an adjunction pair (F, U) : Graphs $\longrightarrow$ Cat between the category of (reflexive, directed)-graphs and the category of categories. The comonad FU : Cat $\longrightarrow$ Cat gives a cosimplicial resolution of C. Let's unpack what we mean by this. Indeed, the object $G(C) := (FU)_\bullet(C)$ lives inside sCat, and each morphism between the levels of the simplicial object is the identity on objects of underlying categories, $G(C)$ can be viewed as an object living inside Cat_Δ, and hence a functor between Cat and Cat_Δ. There is a canonical map $G(C) \longrightarrow C$, where C is viewed with the trivial simplicial structure, and this map is a homotopy equivalence for the model structure given on Cat_Δ. This fact can be can be seen most easily by the fact that the essential image of U consists of UF-projective objects (those with a section $T \longrightarrow UF(T)$). The simplicial set $[n] \longmapsto \mathrm{Hom}_{\mathrm{Cat}_\Delta}(G([n]), C)$ is called the *homotopy coherent nerve* of a simplicial enriched category C, denoted as a functor $N : \mathrm{Cat}_\Delta \longrightarrow$ sSet. The right Kan extension of this functor, denoted $\mathfrak{C}(-)$, is another sort of geometric realisation functor.

1.2. Perhaps a motivation for the above construction could be seen as follows. We start with the question of considering, for a small category C and D a simplicially enriched category, the *homotopy coherent diagrams* in D with shape C. It has been shown that such diagrams are given by the set $\mathrm{Hom}_{\mathrm{Cat}_\Delta}(G(C), D)$. When giving Cat_Δ a model structure called the *Bergner model structure*, it can be see that $G(C) \longrightarrow C$ is a cofibrant resolution. As we have noted, the construction $G(-)$ realises any category C as the homotopy category of the simplicially enriched category $G(C)$.

1.3. If C, D are infinity categories, we can get the simplicial category with morphisms $\mathrm{Fun}(C, D)$, which we noted formed a weak Kan-complex (so was an infinity category itself). Applying the construction above gives the infinity category of infinity categories, which to be precise, is actually an $(\infty, 2)$-category. It is true that the nerve of a simplicial category enriched in Kan complexes (i.e. an infinity groupoid) is an infinity category ([1, Proposition 1.1.5.10]), so called the infinity category of spaces. One gets Cat_∞ by restricting to objects whose mapping spaces are weak Kan complexes.

1.4. Relatedly, if C is a fibrant simplicial category, and x, y are a pair of objects, the co-unit map $u : \mathrm{Map}_{\mathfrak{C}(N(C))}(x, y) \longrightarrow \mathrm{Map}_C(x, y)$ is a weak homotopy equivalence of simpicial sets. This implies that our adjunction above agrees with model structures.

1.5. There is another definition that we may prefer for $\mathfrak{C}([n])$. It is the nerve of the category whose objects are $\{0, 1, ..., n\}$, and for any (i, j), for $j < i$ the morphisms is empty, for $j \geq i$ is $P_{i,j}$, the subset of all posets, with composition equal to union of sets. See [1, Definition 1.1.5.1.] for more details.

1.6. **Straightening and Unstraightening:** *See [1, 2.2.1] for more details.* Fix a simplicial set S, a simplicial category C and a functor $\phi : \mathfrak{C}[S] \longrightarrow C^{op}$. Given an object $X \in \mathrm{sSet}_{/S}$, let v denote the cone point of X^c. Consider the simplicial category $\mathcal{M} := \mathfrak{C}[X^c] \coprod_{\mathfrak{C}[X]} C^{op}$; we get a simplicial functor $\mathrm{St}_\phi(X) : C \longrightarrow \mathrm{sSet}$ described by $\mathrm{St}_\phi X(c) = \mathrm{Map}_{\mathcal{M}}(c, v)$ where $c \in \mathrm{ob}(C)$. Hence, St_ϕ can be viewed as a functor from $\mathrm{sSet}_{/S} \longrightarrow \mathrm{Fun}(C, sSet)$. It is climit preserving, and thus admits a right adjoint by the adjoint functor theorem, called Unstraightening. It is proved in [1,Theorem 2.2.1.2] that this is a Quillen equivalence, for two model structures.

1.7. The relevance for us will be in relation to coCartesian functors. Let us elaborate: Given a functor $F : D \longrightarrow C$ of infinity categories, a co-cartesian fibration is an inner fibration of simplicial sets whose induced map on nerves is a co-cartesian fibration in the usual sense (initial among lifts). Give an example of cartesian fibrations for schemes.

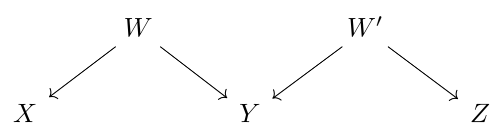
1.8. There is a natural equialence between ∞-categories of functors $C \longrightarrow \mathrm{Cat}_\infty$ and the ∞-categories of coCartesian fibrations. Note the functors from $C \longrightarrow \mathrm{Cat}_\infty$ naturally forms a $(\infty, 2)$-category (the homotopy coherent nerve of a simplicial set may not be an ∞-category!), so we need to restrict to invertible natural transformations.

1.9. **Commutative monoids**
Denote the category $\mathrm{Fin}^{\mathrm{part}}$ to be the category of finite sets with partially defined maps. Then a commutative monoid X in C is defined to be a functor $X : N(\mathrm{Fin}^{\mathrm{part}}) \longrightarrow C$ such that $X(I) \longrightarrow \prod_{i \in I} X(\{i\}) = X(*)^I$ is an isomorphism. The morphism $\emptyset \longrightarrow *$ defines a unit object, and $I \longrightarrow *$ defines the sum maps. In particular, any partially define map $f : I \longrightarrow J$ induces sum maps when looking on fibers $f^{-1}(j) \longrightarrow \{j\}$. It is then clear what a symmetric monoidal object should be. One can then apply this to the Cat_∞.

1.10. We can get another definition of a symmetric monoidal infinity category by utilising straightening/unstraightening. It will be a coCartesian fibration $C^\otimes \longrightarrow \mathrm{Fin}^{\mathrm{part}}$ such that $C^\otimes_I \longrightarrow \prod_{i \in I} C^\otimes_*$ induced by the partially defined maps $I \longrightarrow \{i\}$ is an equivalence. Starting with a symmetric monoidal ∞-category $(C, \otimes)$ in the old sense, $C^\otimes$ is roughly efined as follows. Objects of $C^\otimes$ are pairs $(I, (X_i)_{i \in I})$ of a finite set I and objects $X_i \in C$. A map $(I, (X_i)_{i \in I}) \longrightarrow (J, (Y_j)_{j \in J})$ in $C^{\otimes}$ is given by a partially defined map $f : I \longrightarrow J$ together with maps $\otimes_{i \in f^{-1}(j)} X_j \longrightarrow Y_j$ for all $j \in J$.

1.11. A lax symmetric monoidal functor $C^{\otimes} \longrightarrow D^{\otimes}$ between symmetric monoidal functors fibered over $\mathrm{Fin}^{\mathrm{part}}$ is a functor $F^{\otimes}$ that preserves locally coCartesian lifts of the form $I \longrightarrow \{i\}$. In particular, $F^{\otimes}$ is symmetric monoidal when it induces ismorphisms $\otimes_{i \in I} F(X_i) \longrightarrow F(\otimes_{i \in I} X_i)$.

1.12. **The correspondance category**
Let $(\Delta^n)^2_+$, which consists of the subset spanned by simplices $(i, j) \in \{0, 1, ..., n\}^2$ with $i \geq j$. Varying n gives a cosimplicial category $(\Delta^{\bullet})^2_+$. A correspondence infinity category (C, E) consists of n-simplices whose maps are from $(\Delta^n)^2_+$ and whose arrows going down-right are in E and small squares all small Cartesian. Part of the definition is that C has all finite limits, and E is stable under pullbacks and composition. This insures that the category of correspondences is an infinity category; in particular, filling in the horn:



can be filled in with a fibre product.

1.13.
1.14. **References:**

1. *Higher Topos Theory*, J. Lurie.

2. *On the structure of simplicial categories associated to quasi-categories*, E. Riehl.