0.1. **The join construction**
Given two simplicial sets K and L, we denote the join $K \star L$ by the simplices:

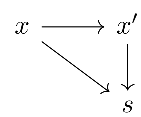
$$(K \star L)_n = K_n \cup L_n \cup_{i+j+1=n} K_i \times L_j.$$

We have $\star$ commutes with colimits is either argument, and $\Delta^i \star \Delta^j \simeq \delta^{i+1+j}$. There is a notion of join in classical category theory, and our construction reduces to it.

0.2. For us, an important construction will be the cone and cocone, $K^{\triangleleft}, K^{\triangleright}$; these are left and right joins with Δ^0, respectively. The join $\Lambda^{2\,\triangleright}_0 \simeq \Delta^1 \times \Delta^1$, for example.

The join $C \star D$ of two infinity categories is again an infinity category.

0.3. **Slice categories**
Suppose we have a infinity category $\mathcal{C}$ and an object of it represented by a morphism $\{s\} \longrightarrow \mathcal{C}$, we want the category of objects over $\{s\}$ to be again an infinity category, but essentially to includes simplices containing s as a vertex; this would be the appropriate generalisation to include the fact that we dont have commutativity anymore when considering diagrams like:



so we need to encode higher simplices. The formal definition for an arbitrary morphism $p : L \longrightarrow \mathcal{C}$, the simplicial set $\mathcal{C}_{/p}$ is characterised by:

$$\mathrm{Mor}_{\mathrm{sSet}}(K, \mathcal{C}_{/p}) \simeq \mathrm{Mor}_p(K \star L, \mathcal{C}).$$

Here, Mor_p is used to indicate morphisms whose restriction to L in the above coincides with p. Using the Yoneda lemma, we can calculate the simplices of this slice category. The other slice category $\mathcal{C}_{p/}$ is defined by maps from $L \star K$.

When $\mathcal{C}$ is an infinity category, $\mathcal{C}_{/p}$ is also an infinity category, but this takes work to prove and I haven't thought about it.

If $p : A \longrightarrow B$ is a functor of usual categories, we have $N(B_{/p}) \simeq N(B)_{/N(p)}$, via the following equality:

$$N(B_{/p})_n = \mathrm{Hom}_{\mathrm{Cat}}([n], B_{/p}) = \mathrm{Hom}_p([n]\star A, B) = \mathrm{Hom}_{N(p)}(N([n]\star A), B) = \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, N(B)_{/N(p)}).$$

0.4. **Final and initial objects**
Given an objects $x \in \mathcal{C}$ of an infinity category, a final object is so such that $\mathcal{C}_{/x} \longrightarrow \mathcal{C}$ is an acyclic fibration of simplicial sets. Some equivalent definitions are:

1. The mapping spaces $\mathrm{Map}_{\mathcal{C}}(x', x)$ are acyclic Kan complexes.
2. Every simplicial sphere $\alpha : \partial\Delta^n \longrightarrow \mathcal{C}$ such that $\alpha(n) = x$ can be filled to the entire n-simplex $\Delta^n \longrightarrow \mathcal{C}$.

The mapping spaces appearing in the above definition is defined to be the fibre product of $x' \to \mathcal{C} \leftarrow \mathcal{C}_{/x}$.

0.5. There is the following useful lemma [1,Theorem 2.1.3.4]: Let $p : S \longrightarrow T$ be a left fibration of simplicial sets. Suppose that for every vertex $t \in S$, the fibre S_t is contractible (acyclic Kan complex). Then p is a trivial Kan fibration (acyclic fibration). Furthermore, if T is a Kan complex, then p is a Kan fibration.

0.6. **Limits and colimits**
If K is a simplicial set and $\mathcal{C}$ is an infinity catgory, then the limit/colimit of a diagram $p : K \longrightarrow \mathcal{C}$ is a final/initial object of $\mathcal{C}_{/p}/\mathcal{C}_{p/}$. In particular, these notions, agree with homotop limits/colimits in the for simpicial categories, but we haven't looked at these. There is also another definition, being that $\mathcal{C}_{p^{\triangleleft}/} \longrightarrow \mathcal{C}_{p/}$ being an acyclic fibration in the case of colimits, which can be used to define relative colimits:

$$C_{p^{\triangleleft}/} \simeq C_{p/} \times_{D_{fp/}} D_{fp^{\triangleleft}/}.$$

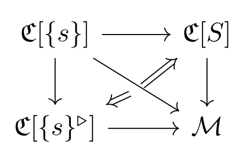
for $f : C \longrightarrow D$ an inner fibration of simplicial sets, is a trivial fibration of simplicial sets. This agrees with the previous definition $D = \{\star\}$. We will use this next week to construct infinity Kan extensions.

0.7. **Straightening and Unstraightening:** *See [1, 2.2.1] for more details.*
Fix a simplicial set S, a simplicial category C and a functor $\phi : \mathfrak{C}[S] \longrightarrow C^{op}$. Given an object $X \in \mathrm{sSet}_{/S}$, let v denote the cone point of $X^{\triangleright}$. Consider the simplicial category $\mathcal{M} := \mathfrak{C}[X^c] \coprod_{\mathfrak{C}[X]} C^{op}$; we get a simplicial functor $\mathrm{St}_\phi(X) : C \longrightarrow \mathrm{sSet}$ described by $\mathrm{St}_\phi X(\triangleright) = \mathrm{Map}_{\mathcal{M}}(c, v)$ where $c \in \mathrm{ob}(C)$. Hence, St_ϕ can be viewed as a functor from $\mathrm{sSet}_{/S} \longrightarrow \mathrm{Fun}(C, sSet)$. It is colimit preserving, and thus admits a right adjoint by the adjoint functor theorem, called Unstraightening. It is proved in [1,Theorem 2.2.1.2] that this is a Quillen equivalence, for two model structures.

0.8. We are grateful that it is possible to associate for every simplicial set X in $\mathrm{sSet}_{/S}$ a corresponing functor $\mathrm{St}_{\mathrm{id}} X : \mathfrak{C}(S)^{\mathrm{op}} \longrightarrow \mathrm{sSet}$; in practise, this is difficult as we will have needed to keep track of many higher coherence data. We defined the unstraightening functor $\mathrm{Un}_{\mathrm{id}} Y$, where $Y \in \mathrm{sSet}^{\mathcal{C}}$ very abstractly, so let us understand it a bit more concretely. Let s be a vertex of S. We have:

$$\mathrm{Hom}_{\mathrm{sSet}_{/S}}(\{s\}, \mathrm{Un}_{\mathrm{id}}(Y)) = \mathrm{Hom}_{\mathrm{sSet}^{\mathcal{C}}}(\mathrm{St}_{\mathrm{id}}\{s\}, Y).$$

Here $\mathcal{C} = \mathfrak{C}[S]^{op}$. Lets consider the pushout:



When trying to understand $\mathcal{M}$ is may be instructive to note that $\mathfrak{C} : \text{sSet} \longrightarrow \text{sCat}$ is colimit preserving and therefore the pushout could be computed on the level of sSet.

We need to understand $\text{Mor}_{\mathcal{M}}(s', v)$ where $s' \in \text{ob}(S)$. The only way there can exist an edge $s'' \longrightarrow v$ for any vertex in S is if we have a triangle $s'' \longrightarrow s \longrightarrow v$, and a two simplex realising composition. Hence a natural transformation in $\text{Hom}_{\text{sSet}^{\mathcal{C}}}(\text{St}_{\text{id}}\{s\}, Y)$ is determined purely by where it sends $\{s\}$ i.e., its determined by points of $Y(s)$.

If $Y(-)$ lands in weak Kan complexes (i.e. infinity categories), then as noted in [1,Theorem 2.2.2.11], the fibre of $\text{Un}_{\text{id}}(Y)$ is literally gives by $Y(s)$. This can be seen by looking at maps $\Delta^n \longrightarrow \text{Un}_{\text{id}}(Y)$, such that we have a factoring $\Delta^n \longrightarrow \{s\} \longrightarrow S$

Performing a similar argument sort $\{s \longrightarrow t\}$ as above, any morphism $s'' \longrightarrow v$ will give us a simplex with vertices s'', s, t, v, and so one sees that functors $\text{Hom}_{\text{sSet}^{\mathcal{C}}}(\text{St}_{\text{id}}\{s \longrightarrow t\}, Y)$ corresponds to a simplicial map $Y(s) \longrightarrow Y(t)$. A commutative diagram $s \longrightarrow t \longrightarrow r$ gets sent to a not necessarily commutative diagram $Y(s) \longrightarrow Y(t) \longrightarrow Y(r)$ with a homotopy between the compositions.

We can thus see how morphisms to the cone gives way to a generalised Grothendieck construction.

0.9. **Cartesian Morphisms** Let $p : X \longrightarrow S$ be an inner fibration of simpicial sets. We shall say $f$is p-Cartesian if the inducted map:

$$X_{/f} \longrightarrow X_{/y} \times_{S_{/p(y)}} S_{p(f)}$$

is a trivial Kan fibration. The definition helps us make sense of 'fibres varying covariantly'.

When $\mathcal{C}$ is an ordinary category, and $p : N(\mathcal{C}) \longrightarrow \Delta^1$ is a map, automatically an inner fibration, then $f : x \longrightarrow Y$ in $\mathcal{M}$ is p-cartesian if and only if it is cartesian in the clasical sense.

0.10. **Cartesian fibrations** We say a map $p : X \longrightarrow S$ os simplicial sets is a Cartesian fibration if p is an inner fibration and for every edge $f : x \longrightarrow y$ of S and every vertex $\widetilde{y}$ of X lifting y, there exists a p-Cartesian arrow $\widetilde{f} : \widetilde{x} \longrightarrow \widetilde{y}$ with $p(\widetilde{f}) = f$. We can also define co-cartesian fibrations similarly.

0.11. The relevance for us will be in relation to coCartesian functors. Suppose we're interested in $\text{Fun}(C, \text{Cat}_\infty)$ of infinity categories. As noted in above, the functors are valued in infinitt categories, unstraightening will gives us a family of simplicial sets over C in an explicit way. It is true that this will end up being a co-cartesian fibration, and indeed there is an equivalece the two infinity categories. The precise equivalence is respect to certain model structures that we will not be going into.

0.12. **Conclusion**

We know a way now of talking about functors $\mathcal{D}_0 : C \longrightarrow \text{Cat}_\infty$. Suppose C is the category of schemes, recall that we're interested in the setting where we have a distinguished class of morphisms E for which functors like $f_!$ are defined. In the previous talk, we constructed the correspondence category $\mathrm{Corr}(C, E)$. In the next talk, we will see how to extend $\mathcal{D}_0$ to a functor $\mathcal{D} : \mathrm{Corr}(C, E) \longrightarrow \mathrm{Cat}_\infty$.

0.13. **References:**

1. *Higher Topos Theory*, J. Lurie.