Grothendieck and anabelian geometry

The 1958 ICM address

THE COHOMOLOGY THEORY OF ABSTRACT ALGEBRAIC VARIETIES

By ALEXANDER GROTHENDIECK

It is less than four years since cohomological methods (i.e. methods of Homological Algebra) were introduced into Algebraic Geometry in Serre's fundamental paper[11], and it seems already certain that they are to overflow this part of mathematics in the coming years, from the foundations up to the most advanced parts. All we can do here is to sketch briefly some of the ideas and results. None of these have been published in their final form, but most of them originated in or were suggested by Serre's paper.

Schemes

Definition (affine scheme)

Given any commutative ring A, define the affine scheme $\operatorname{Spec} A$ to be the locally ringed space consisting of the prime ideals of A equipped with the Zariski topology. The structure sheaf is defined by $O_{\operatorname{Spec} A}(D(f)) = A_f$ on distinguished open sets $D(f)$. A scheme is a locally ringed space where every point has a neighborhood isomorphic to an affine scheme.

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"The very notion of a scheme has a childlike simplicity - so simple, so humble in fact that no one before me had the audacity to take it seriously." – Alexander Grothendieck

Grothendieck at the IHES (1958–1970)

Grothendieck's EGA and SGA

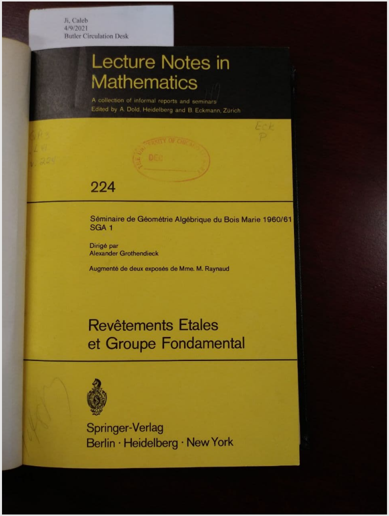
EGA

1. Le langage des schémas
2. Étude globale élémentaire de quelques classes de morphismes
3. Étude cohomologique des faisceaux cohérents
4. Étude locale des schémas et des morphismes de schémas

SGA

1. Revêtements étales et groupe fondamental
2. Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux
3. Schémas en groupes
4. Théorie des topos et cohomologie étale des schémas
5. Cohomologie l-adique et fonctions L
6. Théorie des intersections et théorème de Riemann-Roch
7. Groupes de monodromie en géométrie algébrique

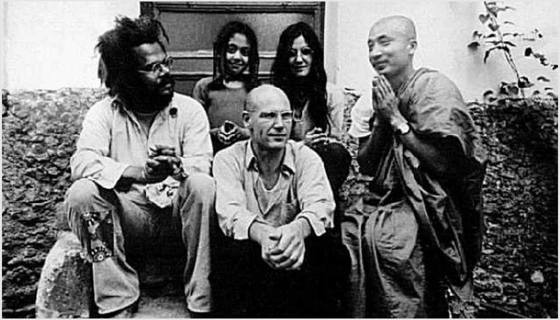
SGA 1: Revêtements étales et groupe fondamental



Develops the theory of étale morphisms, the étale fundamental group, fibered categories, descent, ...

The great turning point, 1970

> *"Yes, it was a* liberation. *And, for the first time in my life I believe, it was then given to me to know the amazed joy and the fullness of one who feels heavy obstacles detaching from him whose existence he had not hitherto even foreseen, and who sees an unsuspected world opening up in front of him, calling him to discover it." – AG, La Clef des Songes*



After

- Founding *Survivre et vivre*
- Boxing police officers
- Being jailed for housing a Japanese Buddhist monk
- Going barefoot in the Canadian winter
- Founding communes, funding agrarian movements
- Awakening his Yin
- Asked to be the leader of Nipponzan Myohoji Buddhism
- Getting his drivers license after nine failures
- Beginning a relationship with a Buddhist nun
- ...

He settles down as a professor at the University of Montpellier.

Polyhedra

- Begins to teach undergraduate math, c. 1975
- 1977, 1978: teaches a course on the cube, another on the icosahedron

The mathematical thought of a child (in so far as it actually leads to a "discovery") could be more "valuable" than a published work (inasmuch as it is mindless and joyless, a routine publication). Or rather, the one is valuable, and the other is spiritual and psychological "junk". ... Polyhedra (take just the cube or even the icosahedron) are an equally inexhaustible source of mathematical reflection and insight on every "level".

A mysterious principle...

From Grothendieck's *Esquisse d'un Programme*:

Whether it happens that such a principle really exists, and even that we succeed in uncovering it from its cloak of fog, or that it recedes as we pursue it and ends up vanishing like a Fata Morgana, I find in it for my part a force of motivation, a rare fascination, perhaps similar to that of dreams. No doubt that following such an unformulated call, the unformulated seeking form, from an elusive glimpse which seems to take pleasure in simultaneously hiding and revealing itself – can only lead far, although no one could predict where...

Grothendieck's challenge

From Grothendieck's *Esquisse d'un Programme*:

The moment seems ripe to rewrite a new version, in modern style, of Klein's classic book on the icosahedron and the other Pythagorean polyhedra. Writing such an exposé on regular 2-polyhedra would be a magnificent opportunity for a young researcher to familiarise himself with the geometry of polyhedra as well as their connections with spherical, Euclidean and hyperbolic geometry and with algebraic curves, and with the language and the basic techniques of modern algebraic geometry. Will there be found one, some day, who will seize this opportunity?

On Belyi's theorem

From Grothendieck's *Esquisse d'un Programme*:

> In the form in which Bielyi states it, his result essentially says that every algebraic curve defined over a number field can be obtained as a covering of the projective line ramified only over the points 0, 1 and ∞. This result seems to have remained more or less unobserved. Yet it appears to me to have considerable importance. To me, its essential message is that there is a profound identity between the combinatorics of finite maps on the one hand, and the geometry of algebraic curves defined over number fields on the other. This deep result, together with the algebraic-geometric interpretation of maps, opens the door onto a new, unexplored world – within reach of all, who pass by without seeing it.

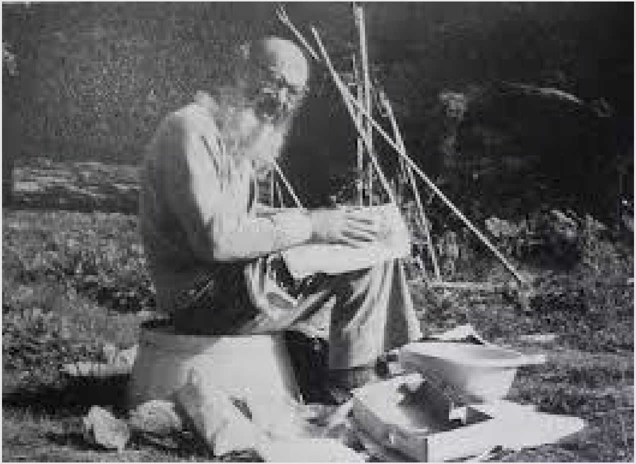
The birth of anabelian geometry

From Grothendieck's *Esquisse d'un Programme*:

ture of this action of $\mathbb{\Gamma}$. One sees immediately that roughly speaking, this action is expressed by a certain "outer" action of $\mathbb{\Gamma}$ on the profinite compactification of the oriented cartographic group $\underline{C}_2^+$, and this action in its turn is deduced by passage to the quotient of the canonical outer action of $\mathbb{\Gamma}$ on the profinite fundamental group $\hat{\pi}_{0,3}$ of $(U_{0,3})_{\overline{\mathbb{Q}}}$, where $U_{0,3}$ denotes the typical curve of genus 0 over the prime field $\mathbb{Q}$, with three points removed. This is how my attention was drawn to what I have since termed "<u>anabelian algebraic geometry</u>", whose starting point was exactly a study (limited for the moment to characteristic zero) of the action of "absolute" Galois groups (particularly the groups $\mathrm{Gal}(\overline{K}/K)$, where K is an extension of finite type of the prime field) on (profinite) geometric fundamental groups of algebraic varieties (defined over K), and more particularly (breaking with a well-established tradition) fundamental groups which are very far from abelian groups (and which for this reason I call "<u>anabelian</u>"). Among

La Gardette

- Spent a year in total solitude at La Gardette, 1979–1980



La Longue Marche à travers la Théorie de Galois

- Written in 1981
- The following quote about it from *Pursuing Stacks*

> I thought it was going to take me a week or two to tour it and kind of recense resources. It took me five months instead of intensive work, and two impressive heaps of notes (baptized "La Longue Marche à travers la théorie de Galois"), to get a first, approximative grasp of some of the main structures and relationships involved. The main emphasis was (still is) on an understanding of the action of profinite Galois-groups (foremost among which $\mathrm{Gal}_{\overline{\mathbb{Q}}/\mathbb{Q}}$ and the subgroups of finite index) on non-commutative profinite fundamental groups, and primarily on fundamental groups of algebraic curves – increasingly too on those of modular varieties (more accurately, modular *multiplicities*) for such curves – the profinite completions of the Teichmüller group. The voyage was the most rewarding and exciting I had in mathematics so far – and still it became very clear that it was just like a first glimpse upon a wholly new landscape – one landscape surely among countless others of a continent unknown, eager to be discovered.

Grothendieck's letter to Faltings

- Anabelian question: How much information about the isomorphism class of the variety X is contained in the knowledge of the étale fundamental group?
- Conjecture (proven by Mochizuki): $\pi_1^{et}(C)$ determines C where C is an appropriate hyperbolic curve.

Gerd Faltings

Shinichi Mochizuki

From Grothendieck's *Esquisse*

a) Combinatorial construction of the Teichmüller tower.

b) Description of the automorphism group of the profinite compactification of this tower, and reflection on a characterisation of $\mathbb{\Gamma} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ as a subgroup of the latter.

c) The "motive machine" $\mathrm{Sl}(2, \mathbb{Z})$ and its variations.

d) The anabelian dictionary, and the fundamental conjecture (which is perhaps not so "out of reach" as all that!). Among the crucial points of this dictionary, I foresee the "profinite paradigm" for the fields $\mathbb{Q}$ (cf. b)), $\mathbb{R}$ and $\mathbb{C}$, for which a plausible formalism remains to be uncovered, as well as a description of the inertia subgroups of $\mathbb{\Gamma}$, via which the passage from characteristic zero to characteristic $p > 0$ begins, and to the absolute ring $\mathbb{Z}$.

e) Fermat's problem.

Why did he stop?

A partial answer, from Grothendieck's *Pursuing Stacks*:

"Doubtless, the very strongest attraction, the greatest fascination goes with the "new world" of anabelian algebraic geometry. It may seem strange that instead, I am indulging in this lengthy digression on homotopical algebra, which is almost wholly irrelevant I feel for the Galois-Teichmüller story. The reason is surely an inner reluctance, an unreadiness to embark upon a long-term voyage, well knowing that it is so enticing that I may well be caught in this game for a number of years – not doing anything else day and night than making love with mathematics, and maybe sleeping and eating now and then."

Questions?