Cohen-Macaulay rings and schemes

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Several of my friends and I were traumatized by Cohen-Macaulay rings in our commutative algebra class. In particular, we did not understand the motivation for the definition, nor what it implied geometrically. The purpose of this paper is to show that the Cohen-Macaulay condition is indeed a fruitful notion in algebraic geometry. First we explain the basic definitions from commutative algebra. Then we give various geometric interpretations of Cohen-Macaulay rings. Finally we touch on some other areas where the Cohen-Macaulay condition shows up: Serre duality and the Upper Bound Theorem.

Contents

1 Definitions and first examples

We begin by listing some relevant foundational results (without commentary, but with a few hints on proofs) of commutative algebra. Then we define depth and Cohen-Macaulay rings and present some basic properties and examples. Most of this section and the next are based on the exposition in [1].

1.1 Preliminary notions

Full details regarding the following standard facts can be found in most commutative algebra textbooks, e.g.

Theorem 1.1 (Nakayama's lemma). *Let $(A, \mathfrak{m})$ be a local ring and let M be a finitely generated A-module. If $M = \mathfrak{m}M$, then $M = 0$.*

Hint. Induct on the number of generators of M. $\square$

Corollary 1.2. *With the conditions above, if any lift of generators of $M/\mathfrak{m}M$ gives generators of M.*

Hint. Let $x_1, \ldots, x_n$ be the lifts to M and apply Nakayama's lemma to $M/(x_1, \ldots, x_n)M$. $\square$

Theorem 1.3 (prime avoidance). *Let $J \subset \cup_{i=1}^n I_i$ be ideals in a ring. If at most two of the I_i are not prime, then $J \subset I_i$ for some i.*

Hint. Induct on n. $\square$

Definition 1.4 (associated primes). Let M be a finitely generated A-module. Then $\mathfrak{p} \in \operatorname{Spec} A$ is an associated prime of M if it is the annihilator of an element of M. The set of these associated primes is denoted $\operatorname{Ass}_A(M)$.

Proposition 1.5. *Let M be a finitely generated nonzero module over a Noetherian ring A. Then*
(i) $\operatorname{Ass}_A M$ is finite and contains all primes minimal over $\operatorname{Ann} M$.
(ii) The union of the $\mathfrak{p} \in \operatorname{Ass}_A M$ is the set of zerodivisors of M.
(iii)

$$\operatorname{Ass}_{A_S} M_S = \{\mathfrak{p} \in \operatorname{Ass}_A M | \mathfrak{p} \cap S = \emptyset\}.$$

(iv) There is a finite filtration of M:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that $M_i/M_{i-1} \cong A_{\mathfrak{p}_i}$, where $\mathfrak{p}_i \in \operatorname{Ass}_A M$.

Hint. One first shows that the maximal elements in the set of annihilators of elements of M are prime ideals, and thus belong to $\operatorname{Ass}_A M$. Then (ii) and (iv) follow easily from this, and moreover we see that all associated primes appear in this way. If $\mathfrak{p}$ is an associated prime of M, then $A/\mathfrak{p} \subset M$, and localizing at S yields (iii). Finally, (i) follows from (iii) after localizing at a given prime minimal over $\operatorname{Ann} M$. $\square$

Theorem 1.6 (Krull's height theorem). *Let $\mathfrak{p} \in \operatorname{Spec} A$ be a prime ideal of a Noetherian ring A that is minimal over an ideal generated by d elements. Then $\operatorname{ht} \mathfrak{p} \leq d$.*

Hint. First consider the case of $d = 1$; this is known has Krull's principal ideal theorem. For this, if $\mathfrak{q} \subsetneq \mathfrak{p}$, show that $A_{\mathfrak{q}}$ has dimension 0 by using constructing a chain of 'symbolic powers' which terminates because $A/(f)$ is Artinian. Then induct for the full statement. $\square$

Definition 1.7 (system of parameters). Let $(A, \mathfrak{m})$ be a Noetherian local ring with Krull dimension d. A system of parameters for A is a set of elements $x_1, \ldots, x_d \in \mathfrak{m}$ such that $\mathfrak{m}$ is a minimal prime over $(x_1, \ldots, x_d)$. A system of parameters exists, and moreover $\mathfrak{m} = \sqrt{(x_1, \ldots, x_d)}$.

Hint. By the height theorem, this cannot be done for less than d elements. Then the existence follows from induction and prime avoidance. $\square$

Proposition 1.8. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Then $\dim A \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2)$.*

Hint. By Nakayama's lemma, a basis of $\mathfrak{m}/\mathfrak{m}^2$ lifts to generators of $\mathfrak{m}$. Now apply Krull's height theorem. $\square$

1.2 Depth and Cohen-Macaulay rings

Definition 1.9 (regular sequence). Let M be an A-module. Then $(x_1, \ldots, x_n) \in A^n$ is a regular sequence on M, or simply an M-sequence, if each x_i is not a zero-divisor of $M/(x_1, \ldots, x_{i-1})M$ and $M(x_1, \ldots, x_n) \neq M$.

Definition 1.10 (depth). Let I be an ideal of A and M a finitely generated A-module such that $IM \neq M$. Then $\operatorname{depth}(I, M)$ is the length of the longest regular sequence on M contained in I.

If $(A, \mathfrak{m})$ is a local ring, then we take the depth of A to be $\operatorname{depth}(\mathfrak{m}, A)$. Depth is a variant of dimension which intuitively measures from big to small. One way to compute depth is through the Koszul complex.

Definition 1.11 (Koszul complex). Let $x_1, \ldots, x_n$ be elements of a ring A. Let $K_1 \cong A^n$ be a free A-module and let $K_p = \wedge^p K_1$. Then the Koszul complex $K(x_1, \ldots, x_n)$ is given by

$$0 \to A \to K_1 \to K_2 \to \cdots \to K_p \to 0.$$

Here each differential is given by $d(a) = (x_1, \ldots, x_n) \wedge a$.

We write $K(x_1, \ldots, x_n) \otimes M$ to denote the complex obtained by tensoring the Koszul complex with M. The cohomology of this complex detects regular sequences.

Theorem 1.12. *Let M be finitely generated over A. Then if*

$$H^i(K(x_1, \ldots, x_n) \otimes M) = 0 \qquad \textit{for } i < r$$

and

$$H^r(K(x_1, \ldots, x_n) \otimes M) \neq 0,$$

then every maximal M-sequence in $I = (x_1, \ldots, x_n)$ has length r.

Proof. [1] Theorem 17.4. □

As a consequence, all maximal M-sequences are of the same length.

We now define Cohen-Macaulay rings.

Definition 1.13. A local ring $(A, \mathfrak{m})$ is Cohen-Macaulay if $\operatorname{depth} A = \dim A$. A ring is Cohen-Macaulay if its localization at all maximal ideals is Cohen-Macaulay.

In general, depth is less than dimension.

Proposition 1.14. *Let $I \subset A$ be an ideal. Then $\operatorname{depth}(I, A) \leq \operatorname{ht} I$.*

Hint. A nonzerodivisor is not contained in any minimal prime, so the result follows by induction. □

In fact, it is equivalent to require localizations at all prime ideals to be Cohen-Macaulay ([1], Prop. 18.8).

2 Geometric properties

2.1 Complete intersections and smoothness

As one might expect, a Cohen-Macaulay scheme is one whose local rings are all Cohen-Macaulay. This gives geometric meaning to the examples we consider.

Proposition 2.1. *A ring A is Cohen-Macaulay if and only if $A[x]$ is Cohen-Macaulay.*

Hint. If A is Cohen-Macaulay, then one notes that every maximal ideal of $A[x]$ is of the form $(\mathfrak{m}, f(x))$, so the depth at each maximal ideal increases by 1. In the other direction, dividing by a nonzerodivisor decreases both the dimension and depth by 1. □

Using this argument, we see that dividing Cohen-Macaulay rings by the ideal generated by a regular sequence preserves the C-M property. These are examples of complete intersections.

Example 2.2. *$k[x,y,z]/(x^2,yz)$ is C-M. However, $k[x,y]/(x^2,xy)$ is not C-M: localizing at (x,y), its dimension is 1 while its depth is 0.*

We will now see that regular local rings are Cohen-Macaulay, so smooth varieties are Cohen-Macaulay.

Proposition 2.3. *Regular local rings are Cohen-Macaulay.*

Hint. Let $(A, \mathfrak{m})$ be a regular local ring with $x_1, \ldots, x_n$ projecting to a basis of $\mathfrak{m}/\mathfrak{m}^2$. Each $A/(x_1, \ldots, x_i)$ is a regular local ring, and by a well-known result ([1], Cor. 10.14) regular local rings are integral domains. □

For example, we see that in the non-example $k[x,y]/(x^2,xy)$, at $(0,0)$ we have $\operatorname{depth} = 0, \dim = 1, \operatorname{emb}\dim = 2$. However, even though $k[x]/(x^2)$ is not smooth, it is Cohen-Macaulay.

2.2 Catenary and equidimensional rings

Recall that associated points are either generic points, which correspond to minimal primes, or are embedded points. In our previous examples, (x) is the generic point and (x,y) is an embedded point in $\operatorname{Spec} k[x,y]/(x^2,xy)$, and (x) is the generic point in $\operatorname{Spec} k[x]/(x^2)$. A key property of local Cohen-Macaulay rings is that all associated primes are minimal. This follows from the following proposition.

Proposition 2.4. *Let M be a finitely generated A-module. If I is an ideal of A containing $\operatorname{Ann}(M)$, then $\operatorname{depth}(I, M) \leq$ the length of any maximal chain of prime ideals descending from a prime containing I to an associated prime of M.*

Proof. [1], Prop. 18.2. □

Proposition 2.5. *Local Cohen-Macaulay rings have no embedded primes.*

Hint. Choose $M = A$ in the previous proposition and use the fact that $\operatorname{depth} A = \dim A$. □

We can go further with the following results. Recall that a ring A is called **universally catenary** if every finitely generated A-algebra is catenary; i.e., all maximal chains between two primes have the same length.

Proposition 2.6. *Cohen-Macaulay rings are universally catenary.*

Hint. Reduce to showing that local C-M rings are catenary, and use the previous proposition. □

A ring is **equidimensional** if all maximal ideals have the same codimension and all minimal primes have the same dimension.

Proposition 2.7. *Local Cohen-Macaulay rings are equidimensional.*

Proof. From the proof of the previous proposition we in fact get that any two maximal chains of prime ideals have the same length. □

Geometrically, this means that a point of a Cohen-Macaulay scheme can't be the intersection of two irreducible subschemes of different dimension.

2.3 The unmixedness theorem and miracle flatness

Let $I = (x_1, \ldots, x_n)$ be an ideal with height n. Then by Krull's height theorem, all minimal primes of I have codimension I. We also have that A/I is Cohen-Macaulay. This leads to the "unmixedness theorem."

Theorem 2.8 (unmixedness theorem). *If A is Cohen-Macaulay, then with I as above every associated prime of A/I is minimal over I.*

This was essentially proven when we showed that Cohen-Macaulay rings have no embedded primes. This theorem can be used to show that a set of polynomials generates the coordinate ring of a projective variety. Indeed, consider the following classical application.

Theorem 2.9 (Max Noether's $AF + BG$ theorem). *Let $f, g \in k[x, y, z]$ be homogeneous polynomials meeting transversely. Then if h vanishes on the intersection, we have $h \in (f, g)$.*

Hint. The transverse condition means that (f, g) has height 2 (and the intersections are reduced points), so we can apply the unmixedness theorem. Let $A = k[x, y, z]/(f, g)$; we have that A has no embedded primes. The key point is that h is in the saturation of (f, g). This implies that (x, y, z) is an associated prime of A, but if it were it would be embedded, contradiction. $\square$

Finally, the "miracle flatness" theorem (also known as Hironaka's criterion) will be used in the applications in the next section. We give the algebraic version, then the geometric version.

Theorem 2.10 (miracle flatness I). *Suppose $\phi : (B, \mathfrak{n}) \to (A, \mathfrak{m})$ is a local homomorphism of Noetherian local rings where A is Cohen-Macaulay, B is regular, and $A/\mathfrak{n}A = A \otimes_B B/\mathfrak{n}$ has pure dimension $\dim A - \dim B$. Then ϕ is flat.*

For example, if A is Cohen-Macaulay and finitely generated over some regular local ring B contained in it, e.g. in the setting of Noether normalization, then it is free over B (flat over local ring $\Rightarrow$ free).

Theorem 2.11 (miracle flatness II). *Suppose $\pi : X \to Y$ is a morphism of equidimensional finite type k-schemes, where X is Cohen-Macaulay, Y is regular, and the fibers of π have dimension $\dim X - \dim Y$. Then π is flat.*

3 Other applications

Here we list two further applications of Cohen-Macaulay rings. Of course there are other applications, like local cohomology and intersection theory, but we do not discuss these. In general, the Cohen-Macaulay condition allows results to be stated cleanly and simply, but of course the general case is always worth studying.

3.1 Serre duality

Let $X \subset \mathbb{P}^N$ be a projective scheme of dimension n and codimension $r = N - n$ and let $\mathcal{F}$ be a coherent sheaf on X. Define the dualizing sheaf

$$\omega_X^\circ = \mathcal{E}xt^r_{\mathbb{P}^N}(\mathcal{O}_X, \omega_{\mathbb{P}^N}).$$

As the notation suggests, the dualizing sheaf is the sheaf of differentials when X is smooth. Also, as we will see in Vakil's proof, this can be interpreted as an adjoint. The defining property of the dualizing sheaf is the existence of a certain isomorphism

$$\mathrm{Hom}_X(\mathcal{F}, \omega_X^\circ) \cong H^n(X, \mathcal{F})^\vee.$$

This is essentially the degree 0 version of Serre duality, and is proven using the *full* version of Serre duality for $\mathbb{P}^n$. We now state Serre duality.

Theorem 3.1. *Let X be a Cohen-Macaulay scheme of dimension n over an algebraically closed field k. Then for $0 \leq i \leq n$ there are natural functorial isomorphisms*

$$\theta^i : \operatorname{Ext}^i(\mathcal{F}, \omega_X^\circ) \to H^{n-i}(X, \mathcal{F})^\vee,$$

where θ^0 is the isomorphism referred to above.

We give a sketch of two proofs of this theorem, one from Hartshorne [2] and one from Vakil [5]. These maps θ^i always exist, and the merit of Hartshorne's proof is that it also shows that if they are isomorphisms, then X is Cohen-Macaulay.

Sketch of Hartshorne's proof.

1. Prove Serre duality for $\mathbb{P}^n$.
2. Prove the defining property of the dualizing sheaf, where $i = 0$.
3. Construct θ^i by showing $\operatorname{Ext}^i(-, \omega_X^\circ)$ is coeffaceable and thus universal.
4. The Cohen-Macaulay condition means that $\operatorname{depth} \mathcal{F}_x = n$. By Auslander-Buchsbaum, we have $\operatorname{pd}_A \mathcal{F}_x = N - n$, so $\mathcal{E}xt^i_{\mathbb{P}^N}(\mathcal{F}, -) = 0$ for $i > N - n$. Using this, we show that $H^i(X, \mathcal{F}(-q)) = 0$ for $q >> 0$.
5. Using this last result, Show $H^{n-i}(X, -)^\vee$ is coeffaceable too, and thus also universal. Thus they are isomorphic to $\operatorname{Ext}^i(-, \omega_X^\circ)$.

□

Sketch of Vakil's proof.

1. Prove the result for $\mathbb{P}^n$.
2. Given $\pi : X \to Y$ and a quasicoherent sheaf on $\mathcal{G}$ on Y, we construct a *right* adjoint $\pi^!$ to π_*, given by $\pi^!\mathcal{G} = \operatorname{Hom}_{\mathcal{O}_Y}(\pi_*\mathcal{O}_X, \mathcal{G})$.
3. By miracle flatness, we have a finite flat map $\pi : X \to \mathbb{P}^n$. Set $\omega_X = \pi^!\omega_Y$.
4. Prove the case $i = 0$.
5. Prove the general statement by showing both sides are coeffaceable. Here we use the condition that π is finite and flat to show that $\pi_*\mathcal{O}_X$ is a vector bundle.

□

Finally, using general results on Ext sheaves we have that, if $\mathcal{F}$ is locally free, then we get

$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X^\circ)^\vee.$$

3.2 The Upper Bound Theorem (combinatorics!)

The Upper Bound Conjecture for spheres was proven by Stanley in [3] using Cohen-Macaualay rings. For the story behind this, see [4].

Let Δ be a simplicial complex on vertices $\{v_1, \ldots, v_n\}$. The f-vector of Δ is simply the vector of the number of faces in each dimension. The h-vector of Δ can be defined as

$$h_0 = 1, \quad h_d = (-1)^d(1 - f_0 + f_1 - \cdots + (-1)^{d-1}f_{d-1}).$$

The question the Upper Bound Conjecture asks is: how large can the f-vector be if Δ is a convex polytope? The conjecture is that if Δ is taken to be a cyclic polytope: the convex hull of n distinct points on the curve $(x, x^2, \ldots, x^d)$, then the maximum for each f_i is reached. An equivalent way to formulate this is by the inequalities

$$h_i \leq \binom{n-d+i-1}{i}, \tag{1}$$

where Δ is the boundary complex of a d-dimensional simplicial convex polytope with n vertices.

Actually, this version was proven by McMullen first, but this statement is generalizable to spheres. Indeed, there are triangulations of spheres which could not be handled with McMullen's approach. Stanley filled this gap using Cohen-Macaulay rings.

Let A_Δ be the coordinate ring

$$A_\Delta := K[v_1, \ldots, v_n]/I,$$

where I consists of the squarefree monomials on the v_i that do not comprise a face of Δ. We can show that the generating function of the Hilbert function of A, multiplied by $(1-x)^d$, gives precisely the h-vector $h_0 + h_1 x + \cdots$. Now suppose A_Δ is Cohen-Macaulay. By miracle flatness, we can then show that the h-vector is an "O-sequence", a simple condition which implies that the desired inequality (1) holds.

It remains to show that A_Δ is Cohen-Macaulay where $|\Delta|$ is a sphere. This follows from a result of Reisner, who showed this condition is equivalent to a certain result regarding the homology of various facets of Δ. This holds for spheres, and thus Stanley proved the Upper Bound Conjecture for spheres.

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