DERIVED ∞-CATEGORIES

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The purpose of this talk is to explain where derived categories and triangulated categories are situated in the modern perspective to a classical algebraic geometer. We will explain where the non-functoriality of cones in triangulated categories comes from and use this to motivate the definition of stable ∞-categories. We will then construct the derived ∞-category. Finally, we will say some words about symmetric monoidal ∞-categories and the Grothendieck construction. All of this comes from Lurie [Lur17], though in the last section we briefly follow Scholze's exposition [Sch22].

1. Triangulated categories and their colimits and cones

We recall that a triangulated category is given by an additive category $\mathcal{D}$ with a set of exact triangles and a translation functor satisfying various properties. The key property is that of a cone: for each $X \xrightarrow{f} Y$, there is an object $C(f)$ such that

$$X \xrightarrow{f} Y \xrightarrow{g} C(f) \xrightarrow{h} X[1]$$

is an exact triangle. One of the main properties of triangulated categories is that they give long exact sequences; for example in the derived category of an abelian category, applying the functor $H^0(-)$ to an exact triangle leads to a long exact sequence in cohomology.

Let us review how the cone is constructed in an important example of triangulated categories: the derived category $D(\mathcal{A})$ of an abelian category $\mathcal{A}$. Given $f : X \to Y$ a map of elements of $D(\mathcal{A})$, we set $C(f) = X[1] \oplus Y$ and define the differential on the complex $C(f)$ by

$$d = \begin{pmatrix} d_{X[1]} & 0 \\ f[1] & d_Y \end{pmatrix}.$$

The triangle maps $Y \to C(f) \to X[1]$ are naturally induced. The cone is unique up to isomorphism. For example, in the case X and Y are given by a complex concentrated in one degree, we could instead define $C(f)$ as $\operatorname{coker} f$, and the two resulting triangles are isomorphic – though not in the homotopy category; this is where inverting quasi-isomorphisms in the derived category comes in!

However, the cone is not unique up to canonical isomorphism; in particular, the cone construction is not functorial, as the following simple example shows.

$$\begin{array}{ccccc} R & \longrightarrow & 0 & \longrightarrow & R[1] \\ \downarrow & & \downarrow & & {\scriptstyle 0}\downarrow{\scriptstyle \mathrm{id}} \\ 0 & \longrightarrow & R & \longrightarrow & R[1] \end{array}$$

The cone is supposed to be interpretable as the colimit of $0 \leftarrow X \to Y$. Thus this issue of the non-functoriality of the cone can be interpreted as a need for homotopy limits and colimits. Grothendieck realized this deficiency of derived and triangulated categories from the beginning, and (on a whim, it seems) wrote a 2000 page manuscript on derivators to solve this problem. Nowadays derivators are seen as a truncation of ∞-categories which may be especially useful for certain problems. However, our goal today is to discuss ∞-categories.

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2. Stable ∞-categories

Recall that the derived category of and abelian category $\mathcal{A}$ is obtained from the homotopy category of $\mathcal{A}$. Morally, it should still be considered as some type of homotopy category. Indeed, the derived ∞-category will be seen to be an underlying ∞-category whose homotopy category recovers the ordinary derived category. Such ∞-categories have features which make them known as stable ∞-categories.

Recall that we take '∞-category' to mean a simplicial set which satisfies the inner Kan conditions. A ∞-category is called pointed if it has a zero object. The other notions for stability take inspiration from stable homotopy theory. Recall that cofibers and fibers can be constructed in the category of ordinary topological spaces. These concepts find a more natural home in spectra, which are the principal objects of study in stable homotopy theory. Recall that fiber and cofiber sequences are the same for spectra. The same is true for stable ∞-categories. In fact, according to Lurie, the category of spectra is the free presentable stable ∞-category on one object. It is also the initial presentably symmetric monoidal ∞-category.

Let $\mathcal{C}$ be a pointed ∞-category. A triangle in $\mathcal{C}$ consists of $X \xrightarrow{f} Y \xrightarrow{g} Z$ with a 2-simplex giving a morphism $h\colon X \to Z$ and a 2-simplex $X \to 0 \to Z$ also involving h.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

Now X is a fiber of g if this is a pullback diagram, and Z is a cofiber of f if it is a pushout diagram.

Definition 2.1. *An ∞-category $\mathcal{C}$ is stable if:*
1. It has a 0 element.
2. Every morphism admits a fiber and a cofiber.
3. Every triangle is a fiber sequence if and only if it is a cofiber sequence.

Recall that the homotopy category of an ∞-category $\mathcal{C}$ is the ordinary 1-category obtained by taking objects to be 0-simplices and morphisms to be 1-simplices up to homotopy. An important result of Lurie that takes quite a bit of work is that the homotopy category of a stable ∞-category is triangulated. In fact, the translation functor and class of triangles can be explicitly defined.

3. The derived ∞-category

The derived ∞-category of an abelian category $\mathcal{A}$ can be constructed with a process of taking chain complexes and then localizing the quasi-isomorphisms. However, we will discuss an alternate method. If $\mathcal{A}$ has enough projectives, we will construct a stable ∞-category $\mathcal{D}(\mathcal{A})$ whose homotopy category is the usual derived category.

First, we want to define the infinity category **Ch**($\mathcal{A}$). Intuitively, 1-simplices are chain morphisms, 2-simplices are chain homotopies, etc. More precisely, we note that the 1-category **Ch**($\mathcal{A}$) is a dg-category by setting

$$\underline{\mathrm{Hom}}(A_\bullet, B_\bullet)_n = \prod \mathrm{Hom}(A_i, B_{i+n}).$$

We can make a simplicial category from a dg-category by using the Dold-Kan correspondence, which is an equivalence between nonnegatively graded chain complexes of abelian groups and simplicial abelian groups. Indeed, applying the truncation functor, we have that the 1-category **Ch**($\mathcal{A}$) is enriched over $\mathbf{Ch}_{\geq 0}(\mathbf{Ab})$, and thus enriched over simplicial abelian groups. Then we take the homotopy coherent nerve to get the ∞-category **Ch**($\mathcal{A}$).

Remark. A better way is to directly take the differential graded nerve from the dg-category.

A key result, which takes some work, is that **Ch**$(\mathcal{A})$ is indeed a stable ∞-category.

When there are enough projectives in an abelian category, one can construct the ordinary derived category by taking the homotopy category of the category of chain complexes of projective objects. Such a construction works here: we define

$$D^-(\mathcal{A}) = \mathcal{N}(\mathbf{Ch}_{>>-\infty}(\mathcal{A})).$$

This is also a stable ∞-category whose suspension functor is given by shifting by 1. Moreover, it has a t-structure whose heart is $\mathcal{A}$, and its homotopy category is the usual derived category.

4. Symmetric monoidal ∞-categories and the Grothendieck construction

Recall that writing down functors $\mathbf{Sch}^{\mathrm{op}} \to \mathbf{Cat}$, like QCoh, is not so straightforward because **Cat** is a 2-category. Grothendieck did this by defining fibered categories, which in this case would be $\mathrm{QCoh} = (S, \mathcal{F})$ with morphisms $(S', \mathcal{F}') \to (S, \mathcal{F})$ given by morphisms $f\colon S \to S'$ and $g\colon f^*\mathcal{F} \to \mathcal{F}'$. Then he showed that Cartesian fibrations over a category C are equivalent to functors $C^{\mathrm{op}} \to \mathbf{Cat}$, which is known as the Grothendieck construction.

In the ∞-scenario, a commutative monoid in an ∞-category C is defined as a functor $X: Fin^{part} \to C$ such that for all finite sets I,

$$X(I) \to \prod_{i \in I} X(\{i\})$$

is an isomorphism. A symmetric monoidal category is a commutative monoid in $\infty - \mathbf{Cat}$, so we need to write down functors $Fin^{part} \to \infty - \mathbf{Cat}$.

Now this can be achieved via the $(\infty, 1)$ version of the Grothendieck construction, which is also known as Lurie's Straightening/Unstraightening. We state it without even explaining the definitions.

Theorem 4.1 (Lurie, Straightening/Unstraightening)**.** *There is a natural equivalence between the ∞-categories of functors $C \to \infty -$ **Cat** and the ∞-category of coCartesian fibrations over C.*

References

[Lur17] Jacob Lurie. Higher algebra. https://www.math.ias.edu/~lurie/papers/HA.pdf, September 18 2017.
[Sch22] Peter Scholze. Six-functor formalisms. https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf, October 2022.