Dold-Kan and simplicial homotopy

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Recall...

Simplicial sets are functors
$$\Delta^{\mathrm{op}} \to \mathbf{Set}.$$
A simplicial set X comes with face and degeneracy morphisms
$$d_i : X_n \to X_{n-1}, \qquad s_i : X_n \to X_{n+1}$$
for $0 \leq i \leq n$. These come from $d^i : [n-1] \to [n]$ skipping i and $s^i : [n+1] \to [n]$ repeating i.

Essentially everything that follows comes from the book *Simplicial Homotopy Theory* by Goerss and Jardine [2].

1 The Dold-Kan corespondence

1.1 Setup

Let $A_\bullet$ be a simplicial abelian group. We define three nonnegatively-graded chain complexes of abelian groups associated to $A_\bullet$.

- The Moore complex, denoted A.

  This is the chain complex
  $$\cdots \to A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \cdots \xrightarrow{A}_1 \partial_1 A_0$$
  where $\partial_n = \sum_{i=0}^{n} (-1)^i d_i$.

- The normalized chain complex, denoted NA.

  Here NA_n is the subgroup $\ker \bigcap_{i=0}^{n-1} \Big(\ker d_i : A_n \to A_{n-1} \Big) \subset A_n$, and the differential is induced from that of the Moore complex:
  $$\cdots \to NA_{n+1} \xrightarrow{(-1)^{n+1} d_{n+1}} NA_n \xrightarrow{(-1)^n d_n} A_{n-1} \to \cdots \to NA_1 \xrightarrow{-d_1} NA_0$$
  so $\partial_n = (-1)^n d_n$.

- A modulo degeneracies, denoted A/DA.

  Here DA_n is the subgroup generated by $\bigcup_{i=0}^{n-1} \Big(\operatorname{im} s_i : A_{n-1} \to A_n \Big) \subset A_n$, and the chain complex is naturally obtained from quotienting each group in the Moore complex by these.

In the opposite direction, given a chain complex C (nonnegative, abelian groups), we can define a simplicial abelian group $\Gamma(C)$ as follows: let

$$\Gamma(C)_n = \bigoplus_{\sigma:[n]\twoheadrightarrow[k]} C_k.$$

We will construct the simplicial structure of $\Gamma(C)$ as part of the proof of Dold-Kan.

Theorem 1.1 (Dold-Kan). *The functor $A_\bullet \mapsto NA$ defines an equivalence between the categories of simplicial abelian groups and nonnegatively graded chain complexes of abelian groups.*

In fact, we will show that $A_\bullet \mapsto NA$ and $C \mapsto \Gamma(C)$ are inverse functors. We prove this in the following steps.

1. Construct the simplicial abelian group structure on $\Gamma(C)$.
2. Remark that $\Gamma(C)/D(\Gamma(C)) \cong C$ where $\Gamma(C)/D(\Gamma(C))$ is viewed as a chain complex.
3. Show the composition
$$NA \xrightarrow{i} A \xrightarrow{p} A/D(A)$$
is an isomorphism of chain complexes.
4. Prove that the natural map
$$\Gamma N(A)_n = \bigoplus_{\sigma:[n]\twoheadrightarrow[k]} NA_k \to A_n$$
is an isomorphism of abelian groups.

The proof of all of these requires nothing more than some diagram-chasing and combinatorics. 3 and 4 are done by induction. 1 and 4 together, along with the fact that $C \mapsto \Gamma(C)$ is a functor (which will follow from our proof of 1) imply that $\Gamma \circ N$ is isomorphic to the identity on simplicial abelian groups. 2 and 3 together imply that $N \circ \Gamma$ imply is isomorphic to the identity on chain complexes.

1.2 Proof

1.2.1 Simplicial structure of $\Gamma(C)$

We need to put a simplicial structure on the data

$$\Gamma(C)_n = \bigoplus_{\sigma:[n]\twoheadrightarrow[k]} C_k.$$

That is, given $\theta : [m] \to [n]$, we need to define the action of $\theta^* : C_k \to \Gamma(C)_m = \bigoplus_{\sigma':[m]\twoheadrightarrow[s]} C_s$ where the domain C_k is indexed by some $\sigma : [n] \to [k]$. For this, we first factor the composition

$$[m] \xrightarrow{\theta} [n] \xrightarrow{\sigma} [k] \quad \text{into} \quad [m] \overset{p}{\twoheadrightarrow} [s] \overset{i}{\hookrightarrow} [k].$$

Then we have a map $i^* : C_k \to C_s$ and a choice of a surjection $p : [m] \twoheadrightarrow [s]$. Taking our choice of σ' to be p, we obtain the desired map

$$C_k \xrightarrow{i^*} (C_s, p) \hookrightarrow \bigoplus_{\sigma':[m]\twoheadrightarrow[s]} C_s.$$

Moreover, these maps are functorial in the sense that $C \mapsto \Gamma(C)$ is indeed a functor from chain complexes to simplicial abelian groups. This is fairly obvious by inspection.

1.2.2 $\Gamma(C)/D(\Gamma(C)) \cong C$

Recall that DA_n consists of the degeneracies; i.e. the subgroup generated by images of the maps $s_i : A_{n-1} \to A_n$. We claim that the images of the maps

$$s_i : \Gamma(C)_{n-1} = \bigoplus_{\sigma':[n-1]\twoheadrightarrow[s]} C_s \to \Gamma(C)_n = \bigoplus_{\sigma:[n]\twoheadrightarrow[k]} C_k$$

generate everything but the case where $n = k$, from which the desired result easily follows. But the image consists precisely of those σ which factor through as $[n] \to [n-1] \to [k]$, as desired.

1.2.3 $NA \cong A/D(A)$

It suffices to show that the natural map $\phi : NA_n \to (A/D(A))_n$ is an isomorphism. We use induction. Recall that $NA_n = \bigcap_{0\le i\le n-1} \ker d_i$. We define

$$N_jA_n = \bigcap_{i\le j} \ker d_i \quad \text{and} \quad D_j(A_n) = \text{span}_{\,i\le j} \operatorname{im} s_i.$$

We induct on j to show that $N_jA_n \cong A_n/D_j(A_n)$. For $j = 0$, we wish to show that if $d_0x = 0$, one simply needs to use the fact that $s^0d^0 = 1 \Rightarrow d_0s_0 = 1$. Then we have a split exact sequence

$$0 \to D_0A_n \to A_n \xrightarrow{x\mapsto x-s_0d_0x} N_0A_n \to 0$$

with splitting $N_0A_n \to A_n$ given by the inclusion.

Now assume we have shown the result for $0, \cdots, j-1$ and also for lesser n. We have the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & N_{j-1}A_{n-1} \arrow[r, "s_j"] \arrow[d, "\cong"] & N_{j-1}A_n \arrow[r, "x\mapsto x-s_kd_kx"] \arrow[d, "\cong"] & N_jA_n \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & A_{n-1}/D_{j-1}(A_{n-1}) \arrow[r, "s_j"] & A_n/D_{j-1}(A_n) \arrow[r] & A_n/D_j(A_n) \arrow[r] & 0
\end{tikzcd}$$

One checks that both sequences are exact. The first two vertical arrows are isomorphisms by the inductive hypothesis, so ϕ is an isomorphism as well.

Remark. It is not hard to show from this that in fact we have a splitting $A_n \cong NA_n \oplus DA_n$.

1.2.4 $\Gamma N(A)_n \cong A_n$

We have a morphism of simplicial abelian groups defined by

$$\Psi : \bigoplus_{\sigma:[n]\twoheadrightarrow[k]} NA_k \to A_n,$$

where $\Psi(NA_k, \sigma) = \sigma^*(NA_k)$. We proceed by induction on n. The base case of $n = 0$ is trivial. Next, assuming an isomorphism for $0, \cdots, n-1$, we see that $\operatorname{im}\Psi \supset DA_n$ because Ψ is a morphism of simplicial abelian groups. Since $\operatorname{im}\Psi \supset NA_n$, by 3 we have that Ψ is surjective.

To show injectivity, suppose $\Psi(x) = 0$ where $x = (x_1, \dots, x_{\binom{n+k-1}{k}})$ where x_1 corresponds to $[n] \twoheadrightarrow [n]$. All the rest corresponding to some $\sigma_i : [n] \twoheadrightarrow [k]$ admit a section $f : [k] \hookrightarrow [n]$ with $k < n$. Then $f^*(x) \in \Gamma(NA)_k$ has identity component (corresponding to $[k] \twoheadrightarrow [k]$) given by x_i. But we have $\Psi f^*(x) = f^*\Psi(x) = 0$, so by the inductive hypothesis $f^*(x) = 0$, so in particular $x_i = 0$. Finally, since $NA_n \to A_n$ is just the inclusion for $[n] \twoheadrightarrow [n]$, we have $x_1 = 0$ as desired.

1.3 Addenda

Proposition 1.2. *The inclusion $NA \to A$ is a homotopy equivalence.*

One can construct an explicit homotopy operator; see [2] Theorem III.2.4.

Having proven Dold-Kan, let us now describe its significance and how it fits into simplicial homotopy theory. In simplicial homotopy theory, we replace topological spaces with simplicial sets, which turn out to be an incredibly useful model. They behave much nicer and we can define analogues of many topological notions under these settings which are easier to work with. One can show that by taking the geometric realization, these analogues give back what we started with for topological spaces. Thus, one obtains natural proofs of important theorems in algebraic topology which are proven in a relatively opaque manner classically.

We will begin by defining Kan complexes, which are the analogue in simplicial sets of Serre fibrations in topological spaces. We give examples, notably of simplicial groups. It turns out that these are the objects one uses to define simplicial homotopy and simplicial homotopy groups. These simplicial homotopy groups are shown to coincide with the ordinary homotopy groups of the geometric realization. All these ideas are best expressed (and further developed) in the language of model categories, which we will only touch briefly on. At this point, our work proving Dold-Kan pays off as we can show that the homotopy groups of a simplicial abelian group conicide with the homology groups of its associated chain complexes. This leads to a slew of important results, such as the representability of ordinary cohomology and the spectral definition of it.

2 Kan complexes

Proposition 2.1. *The realization and singular functors are adjoint:*

$$\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) \cong \mathrm{Hom}_{\mathbf{S}}(X, SY).$$

Proof. We have

$$\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) \cong \varprojlim_{\Delta^n \to X} \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y) \cong \varprojlim_{\Delta^n \to X} \mathrm{Hom}_{\mathbf{S}}(\Delta^n, S(Y)) \cong \mathrm{Hom}_{\mathbf{S}}(X, SY)$$

as desired. □

Recall that the kth horn $\Lambda^n_k \subset \Delta^n$ is the subcomplex generated by all $n-1$-faces $d_j(\iota_n)$ except $d_k(\iota_n)$.

Definition 2.2. *A **Kan fibration** $p : X \to Y$ is a map of simplicial sets such that we can always fill in the dotted arrow $\Delta^n \to X$ making the following diagram commute.*

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Compare this with the definition of Serre fibration, which one may define to be a map $f : T \to U$ of topological spaces with the right lifting property with respect to $|\Lambda^n_k| \to |\Delta^n|$. We see by adjointness that f is a Serre fibration if and only if $S(f) : S(T) \to S(U)$ is a Kan extension. The analogous statement for realizations is only true in one direction and much more difficult: the realization of a Kan complex is a Serre fibration. One can show that $\Delta^n \to *$ is **not** a Kan

extension.
Next, one defines a **Kan complex**, or a **fibrant** simplicial set to be a simplicial set X for which $p : X \to *$ is a Kan fibration. For instance, since $|\Lambda^n_k|$ is a strong deformation retract of $|\Delta^n|$, we have that $S(X)$ is a Kan complex. The simplicial set BG, where G is a groupoid, is also a Kan complex. The following condition is equivalent to being a Kan complex:

> Fo each n-tuple of $(n-1)$-simplices $(y_0, \dots, \hat{y_k}, \dots, y_n)$ of Y with $d_i y_j = d_{j-1} y_i$ if $i < j, i, j \neq k$, there is an n-simplex y such that $d_i y = y_i$.

Proposition 2.3. *Simplicial groups H are fibrant.*

Proof. We use induction. Suppose there is some n-simplex y such that $d_i y = x_i$ where $(x_0, \dots, x_{k-1}, x_{l-1}, x_l, \dots, x_n)$ satisfies $d_i x_j = d_{j-1} x_i$ with $l \geq k+2$. We would like to lower l to $l-1$. This works by choosing

$$y' = s_{l-2}(x_{l-1} d_{l-1} y^{-1}) y.$$

Indeed, one can check that $d_i(x_{l-1} d_{l-1} y^{-1})$ is generically e, and the extra terms are chosen so that when $i = l-1$ we have $d_i y' = x_{l-1}$. □

Remark. Using more model category language, another way to characterize Kan fibrations are the simplicial sets with the right lifting property with respect to all **anodyne extensions**. An anodyne extension is an acyclic cofibration; that is an inclusion (cofibration) which is a weak equivalence. One may also define anodyne extensions as those with the left lifting property with respect to Kan fibrations.

3 Simplicial homotopy

Definition 3.1 (Homotopy of simplicial maps). *If $f, g : K \to X$ are simplicial maps, then there is a homotopy $f \to g$ if there exists a simplicial map $h : K \times \Delta^1 \to X$ such that $h \circ (1, d_1) = f$ and $h \circ (1, d_0) = g$.*

Note that this is **not** an equivalence relation; e.g. two maps $\iota_0, \iota_1 : \Delta^1 \to \Delta^n$ are only homotopic in one direction. However, when the target is *fibrant*, then homotopy is an equivalence relation.

Proposition 3.2. *Let X be fibrant and $L \subset K$ be an inclusion of simplicial sets. Then homotopy of maps $K \to X(\mathrm{rel}\, L)$ is an equivalence relation.*

To prove this, one first shows the result for $K = \Delta^0, L = \emptyset$. That is, homotopy of vertices of a fibrant X is an equivalence relation. This is done by choosing appropriate maps of horns $\Lambda^2_i \to X$. For the general case, one show that the map (of function complexes as simplicial sets)

$$i^* : \mathbf{Hom}(K, X) \to \mathbf{Hom}(L, X)$$

is a fibration. But then homotopy of maps $K \to X(\mathrm{rel}\, L)$ corresponds to homotopy of vertices in the fibers of i^* by the exponential law:

$$\mathrm{Hom}_{\mathbf{S}}(\Delta^1, \mathbf{Hom}(K, X)) = \mathrm{Hom}_{\mathbf{S}}(K \times \Delta^1, X).$$

For complete details, see [2], Corollary I.6.2.

Definition 3.3 (Simplicial homotopy groups). *Let $v \in X_0$ be a vertex of a Kan complex X. Then we define $\pi_n(X, v)$ to be the homotopy classes of maps $\alpha : \Delta^n \to X(\mathrm{rel}\, \partial\Delta^n)$ where $\partial\Delta^n$ gets sent to v.*

How is the group structure defined? Well, given two n-simplices $\alpha, \beta : \Delta^n \to X$, we can construct the following extension

$$\begin{tikzcd} \Lambda^{n+1}_n \arrow[r, "{(v,\ldots,v,\alpha,-,\beta)}"] \arrow[d] & X \\ \Delta^{n+1} \arrow[ur, "\omega"'] & \end{tikzcd}$$

where the vs are also taken to be n-simplices. One can check that $\partial(d_n\omega) = (v, \ldots, v)$. Furthermore the homotopy class of $d_n\omega$ is independent of choices. Thus we may define the group operation by

$$[\alpha] \cdot [\beta] = [d_n\omega].$$

As in the topological case, there is a long exact sequence of homotopy groups associated to a fibration. But the key to comparing simplicial homotopy with ordinary homotopy is the following theorem of Quillen, which is very nontrivial.

Theorem 3.4 (Quillen). *The realization of a Kan fibration is a Serre fibration.*

For a proof, see [2], Theorem I.10.10. Now we define a weak equivalence of simplicial sets to be one that induces an isomorphism on all simplicial homotopy groups. Applying the above theorem to the path loop fibration $\Omega X \to PX \to X$ gives that if X is a Kan complex, then $X \mapsto S|X|$ is a weak equivalence. This leads to the following result.

Proposition 3.5. *If X is a complex with some vertex x, then*

$$\pi_n(X, x) \cong \pi_n(|X|, x).$$

All this is summarized in the following two theorems.

Theorem 3.6. *In the category of simplicial sets, take cofibrations to be inclusions, fibrations to be Kan fibrations, and weak equivalences to be those whose realization is a weak equivalence. Then the category of simplicial sets is a closed model category.*

Theorem 3.7. *The realization and singular functors give a Quillen equivalence between simplicial set and topological spaces.*

4 Connections to cohomology

Given a simplicial set X, we can consider the simplicial abelian group $\mathbb{Z}[X]$ and define the cohomology of X to be that of the Moore complex of $\mathbb{Z}[X]$. Through the Dold-Kan theorem, this will allow us to use our results on simplicial homotopy to study cohomology. The first key result is the following.

Proposition 4.1. *If A is a simplicial abelian group, then*

$$\pi_n(A_\bullet) \cong H_n(NA) \cong H_n(A).$$

Sketch. Indeed, we can see this on the level of sets. Every map $(\Delta^n, \partial\Delta^n) \to (A, 0)$ corresponds to the n-simplices of A that are contained in NA_n and are in the kernel of d_n. The homotopy equivalence classes correspond to being in the image of NA_{n+1}. On the level of groups, note that the group structure on the chain complex corresponds to the group structure on $\pi_n(A)$ defined by the group operation on A. One checks that this conicides with the standard group structure on $\pi_n(A)$ by showing it satisfies an appropriate interchange law. $\square$

This is a really wonderful result. For example, suppose we are interested in the Eilenberg-Maclane space $K(G,n)$ where G is an abelian group. Take the complex $G[n]$ with G in the nth component and zeroes elsewhere. Then $\Gamma(G[n])$ is a simplicial abelian group with only nonzero homotopy group $\pi_n(\Gamma(G[n])) \cong G$, so it is the simplicial version of $K(G,n)$! In particular, taking its geometric realization will give $K(G,n)$. In the case of $G=\mathbb{Z}$, we get the extremely simple $\tilde{\mathbb{Z}}[S^n]$ for $K(\mathbb{Z},n)$. We have the following results.

Theorem 4.2. *Let X be a simplical set and let B be an abelian group. Then there are canonical isomorphisms*

$$[X, K(B,n)] \cong H^n(X,B)$$

for all $n \geq 0$.

Some more work has to be done establishing the model structure of simplicial abelian groups and the nature of the functor $X \mapsto \mathbb{Z}X$. For complete details, see [2], Theorem III.2.19.

We can also compute (co)homology through spectra. Given a spectrum E, define $E_n(X) = \pi_n(E \wedge X)$. Let $(H\mathbb{Z})_n = \tilde{\mathbb{Z}}[S^n]$ be the Eilenberg-Maclane simplicial spectrum.

Theorem 4.3. *There is a natural isomorphism*

$$(H\mathbb{Z})_n(X) \cong \tilde{H}_n(X).$$

Sketch (from Dundas's notes [1]). First, if M is a simplicial abelian group, then it follows from Dold-Kan that $\pi_*(|M|) = H_*(C_*(M))$. So we have

$$\tilde{H}_n(X) \cong \tilde{H}_{n+k}(S^k \wedge X) = H_{n+k}(C_*\tilde{\mathbb{Z}}[S^k \wedge X]) \cong \pi_{n+k}\tilde{\mathbb{Z}}[S^k \wedge X].$$

On the other hand, we have

$$(H\mathbb{Z})_n(X) = \mathrm{colim}_k\, \pi_{n+k}((H\mathbb{Z} \wedge X)_k) = \mathrm{colim}_k\, \pi_{n+k}(\tilde{\mathbb{Z}}[S^k] \wedge X).$$

Now for $k > n$, there is an isomorphism

$$\mathrm{colim}_k\, \pi_{n+k}(\tilde{\mathbb{Z}}[S^k] \wedge X) \cong \pi_{n+k}(\tilde{\mathbb{Z}}[S^k \wedge X]).$$

Modulo this last fact, which is apparently by a stability result, we have a proof. □

References

[1] Bjorn Ian Dundas, *Prerequisistes in Algebraic Topology the Nordfjordeid Summer School on Motivic Homotopy Theory.*

[2] Paul G. Goerss, John F. Jardine, *Simplicial Homotopy Theory.*