Global Monodromy of Kloosterman Sheaves

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Abstract

We give a *very* rough outline of Katz's computation of the monodromy groups of Kloosterman sheaves, which can be found in all its glorious details in [Kat70, Chapter 11].

1 Recollections

Let us recall Deligne's equidistribution theorem.

Theorem 1.1 (Deligne's equidistribution theorem). *Let $\mathcal{F}$ be a lisse sheaf on a smooth geometrically connected curve $U/\mathbb{F}_q$ that is pure of weight 0. Assume that the associated arithmetic and geometric monodromy groups are equal: $G = G_{\mathrm{arith}} = G_{\mathrm{geom}}$. Then the conjugacy classes of the Frobenius elements F_u, for $u \in U$, are equidistributed in a maximal compact subgroup K of G.*

Indeed, each u gives rise to a map $\pi_1(\operatorname{Spec} k) \xrightarrow{u_*} \pi_1(U)$, and u_* lies in a compact subgroup of G as $\mathcal{F}$ is pure of weight 0. (See my previous talk, or better, [Kat70, Chapter 3] for more details.)

We want to apply this theorem to Kloosterman sheaves, which we now recall. The Kloosterman sums, which naturally arise as Fourier transforms of Gauss sums, are given by

$$\mathrm{Kl}(\psi, \chi_1, \dots, \chi_n, b_1, \dots, b_n)(k, a) = \sum_{\prod x_i^{b_i} = a} \psi(x_1 + \cdots + x_n)\chi_1(x_1)\cdots\chi_n(x_n).$$

We constructed Kloosterman sheaves Kl such that

$$\mathrm{tr}(F_{a,k} | \mathrm{Kl}_{\overline{a}}) = (-1)^{n-1}\,\mathrm{Kl}(\psi, \chi_1, \dots, \chi_n, b_1, \dots, b_n)(k, a).$$

In the one-variable case these are constructed by b-translation of tensor product of Kummer and Artin-Schreier sheaves; the general case is obtained via convolution.

These Kloosterman sheaves are lisse and pure of weight $n-1$. Thus to apply Deligne's equidistribution theorem, we need to twist by $\frac{n-1}{2}$. Moreover, we are ultimately interested in the equidistribution of sums of the form

$$\sum_{x_1 \cdots x_n = a} \psi(x_1 + \cdots + x_n),$$

so this corresponds to setting the multiplicative characters and b_i to all be 1. Thus we set

$$\mathcal{F} = \mathrm{Kl}_n(\psi, \mathbf{1})\left(\frac{n-1}{2}\right).$$

By Sophia's talk, once we know the monodromy group of $\mathcal{F}$, Deligne's equidistribution result shows that the Kloosterman angles (obtained from the Kloosterman sums as a varies) are equidistributed in a precise sense. Thus it remains to compute the monodromy group.

2 Statement of the theorem

Theorem 2.1. *[Kat70, Theorem 11.1] The Zariski closure G_{geom} of $\rho(\pi_1^g)$ in $\mathrm{GL}(n)$, as an algebraic group over E_λ, is given for $n \geq 2$ by*

$$G_{\mathrm{geom}} = \begin{cases} \mathrm{Sp}(n) & n \text{ even} \\ \mathrm{SL}(n) & pn \text{ odd} \\ \mathrm{SO}(n) & p = 2, n \text{ odd}, n \neq 7 \\ G_2 & p = 2, n = 7. \end{cases}$$

Before we prove this theorem, we will need to recall some more information about $\mathcal{F}$. These results will not only be used in the proof of the theorem but also to verify that in our scenario the image of π_1^a lies in G_{geom}, which is a condition we need to apply Deligne's equidistribution theorem in the first place. Translating the results we have (supposedly) proved for the Kloosterman sheaves to $\mathcal{F}$, we have the following facts [Kat70, 11.0.2].

1. $\mathcal{F}$ is lisse of rank n and pure of weight 0.
2. $\mathcal{F}$ is unipotent as an I_0-representation, with a single Jordan block.
3. $\mathcal{F}$ is totally wild at ∞, with $\mathrm{Swan}_\infty(\mathcal{F}) = 1$.
4. $\det \mathcal{F}$ is trivial.
5. If n is even, there exists an alternating perfect pairing $\mathcal{F} \otimes \mathcal{F} \to \overline{\mathbb{Q}_p}$. If n is odd, there exists a symmetric perfect pairing of that form.

Fact 4 implies that the image of π_1^a is contained in $\mathrm{SL}(n)$. This combined with the alternating/symmetric pairing in Fact 5 implies that the image of π_1^a is indeed contained in G_{geom} as G_{geom} is shown to be as big as possible – except in the $p = 2, n = 7$ case, which is dispatched by an ad hoc argument.

3 Rough outline of the proof

In our situation, we have our mondoromy group $G \subset \mathrm{SL}(V)$ where we take V to be an n-dimensional vector space over (as we may base change) $\overline{\mathbb{Q}_p}$. G contains U_0, a unipotent element with asingle Jordan block which is the image of a topological generator of I_0^{tame}. It also contains $\Gamma_\infty = \rho(I_\infty)$, which is finite. One can then prove that its identity component G^0 is semisimple and acts irreducibly on V.

Passing to the Lie algebra $\mathcal{G}_0$ of G^0, one finds that $\mathcal{G}_0$ is a simple Lie algebra that acts irreducibly on V in which $N = \log(U_0)$ acts nilpotently with a single Jordan block. One then proves the following classification theorem.

Theorem 3.1. *[Kat70, Theorem 11.6] In the situation above, the pair $(\mathfrak{g}_0, \rho)$ is isomorphic to one of the following.*

1. $\mathfrak{g}_0 = \mathfrak{sl}(2), \rho = \mathrm{Sym}^{n-1}(\mathrm{std})$
2. $\mathfrak{g}_0 = \mathfrak{sl}(n), n \geq 3, \rho = \mathrm{std}$
3. $\mathfrak{g}_0 = \mathfrak{sp}(n), n \geq 4$ *even,* $\rho = \mathrm{std}$
4. $\mathfrak{g}_0 = \mathfrak{so}(n), n \geq 5$ *odd,* $\rho = \mathrm{std}$
5. $\mathfrak{g}_0 = \mathrm{Lie}(G_2)$ *if* $n = 7$, $\rho =$ *its unique 7-D irrep.*

From this we can deduce most of the main theorem 2.1, except that in the case of $p = 2, n = 7$, we don't know if G is G_2 or $\mathrm{SO}(7)$. (This case is irrelevant for the equidistribution result addressed last week.) One might guess that it is $\mathrm{SO}(7)$ since in the other $p = 2, n$ odd cases it is $\mathrm{SO}(n)$. But it is in fact G_2. We will discuss this a bit in the next section, but for now we will say something about how this classification theorem is proven.

We would like to classify pairs $(\mathfrak{g}, \rho)$ of faithful irreps into $\mathfrak{sl}(n)$ of finite-dimensional simple Lie algebras over $\overline{\mathbb{Q}_p}$, such that there is some nilpotent $N \in \mathfrak{g}$ with $\rho(N)$ consisting of a single Jordan block.

Fix a Borel subalgebra $\mathfrak{b}$ with nilpotent elements $\mathfrak{n}$ and Cartan subalgebra $\mathfrak{h}$ in $\mathcal{G}$. This defines a set of positive roots. Recall that for each positive root, one can define an $\mathfrak{sl}(2)-$triple $(X_\alpha, H_\alpha, X_{-\alpha})$ with $H_\alpha \in [g_\alpha, g_{-\alpha}]$ and $\alpha(H_\alpha) = 2$.

The hypothesis that $\rho(N)$ has a single Jordan block for $N \in \mathfrak{n}$ is Zariski open, as it is equivalent to some $(n-1) \times (n-1)$ minor being invertible. Furthermore, all endomorphisms $\mathrm{ad}(N')$ (with $N' \in \mathfrak{n}$) have at least $\dim(\mathfrak{h})$ Jordan blocks, and the subset with exactly $\dim(\mathfrak{h})$ Jordan blocks are known as principal nilpotent elements and these are known to be Zariski open in $\mathfrak{n}$. Thus under our hypothesis, we may pick $N_1 \in \mathfrak{n}$ to be principal such that $\rho(N_1)$ has a single Jordan block.

By the Jacobson-Morozov theorem, N_1 can be completed to an $\mathfrak{sl}(2)$-triple. The restriction of ρ to it is irreducible as the image of "e" has a single Jordan block. By a result on the conjugacy of principal $\mathfrak{sl}(2)$-triples (and other results), we obtain that the restriction of ρ to any $\mathfrak{sl}(2)$-triple with $h = h^0$ being the unique element with $\alpha(h) = 2$ for all $\alpha \in B$ is irreducible.

Next, recall that the irreducible representations of $\mathfrak{g}$ are naturally indexed by dominant weights w. For such an irrep V_w, for an $\mathfrak{sl}(2)$-triple described above, one shows that the highest weight of the composite $\mathfrak{sl}(2)$ representation is indeed $w(h^0)$. Combining everything (and other facts) we obtain that if w is dominant, then

$$\dim(V_w) - 1 \geq w(h^0)$$

with equality if and only if the restriction of V_w to every principal $\mathfrak{sl}(2)$-triple is irreducible. (which occurs in our scenario)

We may assume $g \neq \mathfrak{sl}(2)$ as otherwise there is nothing to prove. Using the Weyl dimension formula, one shows that if indeed $\dim(V_w) - 1 = w(h^0)$, then w is actually a fundamental weight.

Finally, we must check all the simple Lie algebras to see which have fundamental weights whch satisfy $\dim(V_w) - 1 = w(h^0)$. Doing this, which takes quite a bit of work, finishes the proof of the classification theorem.

4 The G_2 theorem

In the case that $p = 2$ and $n = 7$, Katz proves that $G_{\mathrm{geom}} = G_2$ by proving it cannot be $SO(7)$. First, he reduces to the case $q = 2$. So assume for sake of contradiction that it is $SO(7)$. Then let

$$\mathcal{G} = \Lambda^3(\mathcal{F}).$$

It is known by Bourbaki that $\Lambda^3(\mathrm{std})$ (of $SO(7)$) is irreducible. Thus it has no invariants or coinvariants, and

$$H^0(\mathbb{G}_m \otimes \overline{\mathbb{F}_2}, \mathcal{G}) = H^2_c(\mathbb{G}_m \otimes \overline{\mathbb{F}_2}, \mathcal{G}) = 0.$$

Assuming this and using previously established information about $\mathrm{Kl}_7(\psi)$ as an I_0-representation, it is possible to show that

$$\mathcal{G}^{I_0} \cong H^1_c(\mathbb{G}_m \otimes \overline{\mathbb{F}_2}, \mathcal{G})$$

has dimension 5, with F_0-eigenvalues $q^{-6}, q^{-4}, q^{-3}, q^{-2}, 1$. (This takes quite a bit of work.) Making the twist $\Lambda^3(\mathrm{Kl}_7(\psi)) = \mathcal{G}(-9)$, we see that these eigenvalues are q^3, q^5, q^6, q^7, q^9. Also, H_c^0 and H_c^2 are still 0. Next we apply the Grothendieck-Lefschetz trace formula to $\Lambda^3(\mathrm{Kl}_7(\psi))$ on $\mathbb{G}_m \otimes \mathbb{F}_2$, which just has the rational point 1. So, negative of the trace of Frobenius at 1 of this sheaf is $q^3 + q^5 + q^6 + q^7 + q^9$.

On the other hand, recall that $\det(1 - TF|\mathcal{F})$ is given by $\sum(-1)^n \operatorname{tr}(F|\Lambda^i\mathcal{F})T^i$. In our scenario, the LHS is also $\exp(-\sum_{n\geq 1} \frac{S_n T^n}{n})$, where the S_n are the traces of $(F_1)^n$ on $\mathrm{Kl}_7(\psi)$, which are integers. This implies that we have

$$\frac{S_1 S_2}{2} - \frac{S_3}{3} - \frac{S_1^3}{6} = q^3 + q^5 + q^6 + q^7 + q^9.$$

Working $\pmod 7$, one can compute the residues of S_1, S_2, S_3 to be $-1, 1, -1$ respectively. For example, since

$$S_n = \sum_{x_1 \cdots x_7 = 1, x_i \in \mathbb{F}_{2^n}} (\psi \circ \operatorname{tr}_{\mathbb{F}_{2^n}/\mathbb{F}_2})(\sum x_i),$$

for $n = 1$ the cyclic shifts by $\mathbb{Z}/7\mathbb{Z}$ shows that

$$S_1 \equiv \sum_{x=1} \psi(7x) = \psi(7) = (-1)^7 \equiv -1 \pmod 7.$$

Since $q = 2$, this gives a contradiction. This shows that $G_{\mathrm{geom}} \neq \mathrm{SO}(7)$, and thus it must be G_2 by the classification theorem.

References

[Kat70] N. M. Katz. *Gauss Sums, Kloosterman Sums, and Monodromy Groups*, volume 116 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1970.