The Atiyah-Hirzebruch Spectral Sequence

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This paper is an expository account of the Atiyah-Hirzebruch spectral sequence, which relates singular cohomology to generalized cohomology theories. In Section 1, we state the spectral sequence and make some remarks on variants of it. In Section 2, we construct the spectral sequence. In Section 3, we apply it to K-theory and see how Bott periodicity simplifies it. Then we use it along with other methods to compute the K-groups of various spaces. Some of these examples will enhance our understanding of the differentials of the Atiyah-Hirzebruch spectral sequence.

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1 Statement of the AHSS

1.1 Generalized cohomology theories

We begin by recalling the definition of reduced generalized cohomology theories.

Definition 1.1. *A* ***reduced generalized cohomology theory*** *is a functor $\tilde{E}^*$ from pointed spaces to graded abelian groups satisfying the following properties.*

1. *$\tilde{E}^*$ is homotopy invariant.*
2. *There is a natural isomorphism $\tilde{E}^* X \cong \tilde{E}^{*+1}\Sigma X$.*
3. *If $A \hookrightarrow X$ is an inclusion of pointed spaces, there's an exact sequence*

$$\tilde{E}^*(X/A) \to \tilde{E}^*(X) \to \tilde{E}^*(A)$$

4. *$\tilde{E}^*$ takes coproducts to products.*

Given an inclusion $A \hookrightarrow X$, we can continue it to a cofiber sequence:

$$A \hookrightarrow X \to X/A \cong X \cup CA \to \Sigma A \to \Sigma X \to \cdots .$$

By applying the cohomology theory $\tilde{E}^*$, the given properties yield the following long exact sequence.

$$\cdots \to E^{n-1}(A) \to E^n(X/A) \to E^n(X) \to E^n(A) \to E^{n+1}(X/A) \to \cdots .$$

There is a slightly different set of axioms for **generalized cohomology theories**. Note that by property 3, the reduced cohomology of a point is trivial. Generalized cohomology theories, which are defined on pairs, do not satisfy this. A key example we will be interested in is K-theory; K-theory is a generalized cohomology theory and reduced K-theory is a reduced generalized cohomology theory. One goes between the two in the following way.

Proposition 1.2. *Given an unreduced cohomology theory E^*, we obtain a reduced cohomology theory $\tilde{E}^*$ by setting*

$$\tilde{E}^*(X, x) \coloneqq E^*(X, \{x\}).$$

In the other direction, we can define

$$E^*(X, A) \coloneqq \tilde{E}^*(X_+ \cup C(A_+)).$$

Note that in the first definition, (X, x) is taken as a pointed space and $(X, \{x\})$ is taken as a pair of spaces. For proofs, see the nLab [9]. In practice, the consequences of this we will use are that $E^*(X, A) = \tilde{E}^*(X/A)$ and that $\tilde{E}^*(X) = \ker(E^*(X) \to E^*(\mathrm{pt}))$.

The following lemma asserts that generalized cohomology theories behave like ordinary cohomology when it comes to maps of spheres.

Lemma 1.3. *Let $\tilde{E}^*$ be a generalized cohomology theory and let $f : S^n \to S^n$ be a continuous map. Then the induced map*

$$f^* : \tilde{E}^*(S^n) \to \tilde{E}^*(S^n)$$

is multiplication by $\deg(f)$.

Proof. Recall that the homotopy class of f is completely determined by its degree. Thus for each k, it suffices to prove the result for a single degree k map. For k positive, first collapse the complement of k open balls to a point. Then apply the identity map to each of the k spheres. This gives a composition

$$S^p \to \vee_{i=1}^k S^p \to S^p$$

of degree k. Then applying E^* and using property 4 shows that f^* is multiplication by k. Using this result and composing with reflections yields the result for negative k as well. □

1.2 Stating the AHSS

The goal of the final page

Let X be a finite CW-complex and let G^* be a generalized cohomology theory[1]. Let $\tilde{G}^* = \ker(G^*(X) \to G^*(\mathrm{pt}))$ be the corresponding reduced theory. The natural topological filtration of X given by its CW-structure

$$\mathrm{pt} = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^n = X$$

[1] The more common notation is E^*, but we will also be using this for the pages of the spectral sequence...

induces a filtration on $G^k(X)$:

$$0 = \tilde{G}^k_n(X) \subset \tilde{G}^k_{n-1}(X) \subset \cdots \subset \tilde{G}^k_0(X) \subset \tilde{G}^k_{-1}(X) = \tilde{G}^k(X),$$

where

$$\tilde{G}^k_p(x) := \ker[\tilde{G}^k(X) \to \tilde{G}^k(X^r)].$$

Oftentimes, to make the fact that we are taking a filtration more evident, this is written as

$$0 = F_n\tilde{G}^k(X) \subset F_{n-1}\tilde{G}^k(X) \subset \cdots \subset F_0\tilde{G}^k(X) \subset F_{-1}\tilde{G}^k(X) = \tilde{G}^k(X).$$

The spectral sequence we will construct will give us each quotient

$$\tilde{G}^k_{p-1}(X)/\tilde{G}^k_p(X) = F_{p-1}\tilde{G}^k(X)/F_p\tilde{G}^k(X)$$

as the entry $E_\infty^{p,k-p}$ on the final page which the spectral sequence converges to.

Page 1

Intuitively speaking, we obtained the terms above by taking successive quotients of a filtration of $\tilde{G}^n(X)$. The spectral sequence begins on the first page by essentially doing these steps in reverse. Namely, we define

$$E_1^{r,n-r} := G^n(X_r, X_{r-1}) \cong \tilde{G}^n(X_r/X_{r-1}).$$

Now X_r/X_{r-1} is nothing but a wedge of S^r's, and generalized cohomology theories send wedges to direct sums. Furthermore, the suspension isomorphism allows us to identify $\tilde{G}^n(S^r)$ with $\tilde{G}^{k-p}(*)$. This gives us the equality

$$E_1^{p,k-p} = \tilde{G}^k(X_p/X_{p-1}) = \bigoplus_{p-\text{cells}} \tilde{G}^k(S^p) = \bigoplus_{p-\text{cells}} \tilde{G}^{k-p}(S^0) = \bigoplus_{r-\text{cells}} G^{k-p}(\text{pt}) = C^p(X, G^{k-p}(\text{pt})).$$

Statement of the spectral sequence

Having defined and worked out the first page, we are now ready to state **Atiyah-Hirzebruch spectral sequence**.

Theorem 1.4. *[Atiyah-Hirzebruch spectral sequence] Let X be a finite CW-complex and let G^* be a generalized cohomology theory with reduced version $\tilde{G}^*$. Then there is a spectral sequence with pages $E_k^{p,q}$ satisfying the following properties.*

- $E_1^{p,q} = C^p(X; G^q(*))$.
- $E_2^{p,q} = \tilde{H}^p(X; G^q(*))$.
- $E_\infty^{p,q} = \dfrac{\ker(\tilde{G}^{p+q}(X) \to \tilde{G}^{p+q}(X^{p-1}))}{\ker(\tilde{G}^{p+q}(X) \to \tilde{G}^{p+q}(X^p))}$.

Remark. By reversing the arrows, one obtains a similar statement for homology.

Remark. According to Adams, this was "probably first invented by G. W. Whitehead" [1]; they became folklore, and was first published by Atiyah and Hirzebruch [2].

1.3 Variants and extensions

Let us comment on some further aspects of this spectral sequence. First, as stated the Atiyah-Hirzebruch spectral sequence gives no information about the multiplicative structure of a generalized cohomology theoy, if it exists at all. For instance, when applied to K-theory it does not give us the ring structure. Another generalized cohomology theory with a rings tructure is given by bordism. In the case of the oriented bordism ring MSO_*, Gray figured out how to determine the ring structure with the Atiyah-Hirzebruch sequence in [6].

As Atiyah and Hirzebruch pointed out in the original paper [2], the Atiyah-Hirzebruch spectral sequence can also be generalized to fiber bundles $F \hookrightarrow Y \to X$. In this case, the spectral sequence is given by local coefficients:

$$E_2^{p,q} \cong H^p(X; K^q(F)) \Rightarrow K^{p+q}(Y).$$

Considering the trivial fibration $Y = X$ yields the original spectral sequence. Moreover, by using singlar cohomology for the generalized cohomology theory, one obtains the Serre spectral sequence.

Finally, we remark that there is an algebraic version of this sequence that relates motivic cohomology to algebraic K-theory. Even the definition of these two theories is rather involved, so it should not be surprising that this version of the spectral sequence, stated below, is significantly more difficult.

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) = \mathrm{CH}^{-q}(X, -p-q) \Rightarrow K_{-p-q}(X).$$

This was first established in 2002 by Suslin and Friedlander [5], building on earlier work by Block and Lichtenbaum. For an introduction to motivic cohomology, we recommend Voevodsky's lectures [10]. For a very distilled account, one may consult the author's slides.

2 Construction of the AHSS

The goal of this subsection is to prove Theorem 1.4. We have already shown how the E_1 page is constructed. It remains to define the differentials and prove the claimed statements about E_2 and E_∞.

2.1 Exact couples

We begin by reviewing the formalism of exact couples, which streamlines the computations.

Definition 2.1. *An* ***exact couple*** *is an exact triangle of abelian groups of the following form.*

$$\begin{array}{ccc} D_1 & \xrightarrow{\alpha} & D_1 \\ {\scriptstyle\gamma}\nwarrow & & \swarrow{\scriptstyle\beta_1} \\ & E_1 & \end{array}$$

One should think of E_1 has the first page of a spectral sequence. Let $d_1 = \beta_1 \circ \gamma$. By replacing E_1 with $E_2 := H(E_1, d_1)$, D_1 with $D_2 := \mathrm{im}(\alpha)$, and β_1 with $\beta_2 = \beta_1 \circ \alpha^{-1}$, it is readily checked that

$$\begin{array}{ccc} D_2 & \xrightarrow{\alpha} & D_2 \\ {\scriptstyle\gamma}\nwarrow & & \swarrow{\scriptstyle\beta_2} \\ & E_2 & \end{array}$$

forms another exact couple, known as the *derived* couple of the first one. Then E_2 is the second page of the spectral sequence, and we can iterate this process to obtain the remaining pages.

While this approach may seem quite abstract, it arises very naturally in the situation we are considering. The basic idea is that our filtration

$$\cdots \subset X^{p-1} \subset X^p \subset X^{p+1} \subset \cdots$$

contains a lot of exact sequences that can be put into an exact couple. Indeed, begin by setting $D_1 = \bigoplus_{p,q} \tilde{G}^{p+q}(X^p)$ and $E_1 = \bigoplus_{p,q} \tilde{G}^{p+q}(X^p/X^{p-1}) \cong C^p(X; G^q(*))$. For each p, consider the exact sequence

$$\cdots \xrightarrow{\beta_1} G^{p+q}(X^p, X^{p-1}) \xrightarrow{\gamma} G^{p+q}(X^p) \xrightarrow{\alpha} G^{p+q}(X^{p-1}) \xrightarrow{\beta_1} G^{p+q+1}(X^p, X^{p-1}) \xrightarrow{\gamma} \cdots.$$

We see that we have indeed created an exact couple, with the initial maps simply being the direct sum of all the long exact sequences associated to the pairs in the filtration. We are now ready to compute the second page.

2.2 Page 2

Before working out what the second page E_2 looks like, we recall that we have $E_1^{p,q} = C^p(X; G^q(*))$ and we want to show that $E_2^{p,q} = H^p(X; G^q(*))$. This suggests that d_2 should be the cellular differential, and this is indeed what we will prove. The key point of this computations is Lemma 1.3, and the rest is (somewhat involved) diagram chasing.

We recall that we have defined $d_1 = \beta_1 \circ \gamma$. To be explicit, we write this out.

$$\cdots \xrightarrow{\beta_1} \tilde{G}^{p+q-1}(X^p/X^{p-1}) \xrightarrow{\gamma} \tilde{G}^{p+q-1}(X^p) \xrightarrow{\alpha} \tilde{G}^{p+q-1}(X^{p-1}) \xrightarrow{\beta_1} \tilde{G}^{p+q}(X^p/X^{p-1}) \xrightarrow{\gamma} \cdots$$
$$\cdots \xrightarrow{\beta_1} \tilde{G}^{p+q-1}(X^{p+1}/X^p) \xrightarrow{\gamma} \tilde{G}^{p+q-1}(X^{p+1}) \xrightarrow{\alpha} \tilde{G}^{p+q-1}(X^p) \xrightarrow{\beta_1} \tilde{G}^{p+q}(X^{p+1}/X^p) \xrightarrow{\gamma} \cdots.$$

Let D_i^{p+1} and D_j^p be the discs associated to any $p+1$-cell and p-cell. Recall that the matrix of the cellular differential corresponding to these two cells is computed as the degree d_{ij} of the composition

$$f_{ij} : \partial D_i^{p+1} \xrightarrow{\phi_i} X^p \xrightarrow{\pi_j} S_j^p,$$

where the second map is projection obtained by quotienting outside the p-cell. We will actually factor this map and rename it as follows.

$$f_{ij} : \partial D_i^{p+1} \xrightarrow{\phi_i} X^p \xrightarrow{\pi} X^p/X^{p-1} \xrightarrow{g_j} D_j^p/\partial D_j^p = S_j^p.$$

Our goal is to show that the ij component of the map

$$\tilde{G}^{p+q-1}(X^p/X^{p-1}) \xrightarrow{\gamma} \tilde{G}^{p+q-1}(X^p) \xrightarrow{\beta_1} \tilde{G}^{p+q}(X^{p+1}/X^p)$$

is indeed $\deg(f_{ij})$. By Lemma 1.3, it suffices to show that the ij component of $\beta_1 \circ \gamma$ is $\phi_i^* \circ \pi^* \circ g_j^*$. Let h_i denote the inclusion $D_i^{p+1}/\partial D_i^{p+1} \hookrightarrow X^{p+1}/X^p$. Then we have the following commutative diagram.

$$\begin{array}{ccccc}
\tilde{G}^{p+q-1}(D_j^p/\partial D_j^p) & & & & \\
\downarrow g_j^* & & & & \\
\tilde{G}^{p+q-1}(X^p/X^{p-1}) & \xrightarrow{\gamma=\pi^*} & \tilde{G}^{p+q-1}(X^p) & \xrightarrow{\beta_1} & \tilde{G}^{p+q}(X^{p+1}/X^p) \\
 & & & \searrow^{\phi_i^*} & \downarrow h_i^* \\
 & & & & \tilde{G}^{p+q-1}(\partial D_i^{p+1}) \cong \tilde{G}^{p+q}(D_i^{p+1}/\partial D_i^{p+1})
\end{array}$$

Here, the horizontal arrows compose to d_1, while the vertical arrows pick out the cells corresponding to D_i^{p+1} and D_j^p. The commutativity of the lower triangle follows from the naturality of the long exact sequences (recall that the isomorphism $\tilde{G}^{n+1} \circ \Sigma \cong \tilde{G}^n$ forms the long exact sequence!). Thus we see that the ij component of $d_1 = \beta_1 \circ \gamma$ is indeed $\phi_i^* \circ \pi^* \circ g_j^*$, so by Lemma 1.3 we have that d_1 coincides with the cellular differential, as desired.

2.3 Page n and convergence

It remains to prove the convergence of our spectral sequence. For convenience, we recall the goal.

$$E_\infty^{p,q} \stackrel{?}{=} \frac{F_{r-1}\tilde{G}^n(X)}{F_r\tilde{G}^n(X)} := \frac{\ker[\tilde{G}^n(X) \to \tilde{G}^n(X^{r-1})]}{\ker[\tilde{G}^n(X) \to \tilde{G}^n(X^r)]}.$$

We begin by proving the following general statement about spectral sequences from exact couples.

Proposition 2.2. *In the exact couple given by Definition 2.1, for all $n \geq 1$ we have*

$$E_{n+1} = \frac{\gamma^{-1}(\alpha^n D_1)}{\beta_1(\ker \alpha^n)}$$

Proof. The proof is simply checking the definitions. For $n = 1$, we have $d_1 = \beta_1 \circ \gamma : E_1 \to E_1$. Since $\operatorname{im} \gamma = \ker \alpha$, this means that $\ker d_1 = \gamma^{-1}(\alpha D)$. Similarly, $\operatorname{im} d_1 = \beta_1(\ker \alpha)$. Thus

$$E_2 := H(E_1, d_1) = \frac{\gamma^{-1}(\alpha D_1)}{\beta_1(\ker \alpha)}.$$

Then one repeats this argument (with a little more care) for the inductive step going from E_n to E_{n+1}. □

In our case of an n-dimensional CW-complex, the filtration stabilizes at X^n. Then for $p > n$, the (p, q) position of E_1 consists of $G^{p+q}(X^n, X^n) = 0$. Then for degree reasons, $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ is trivial for $r > n$. Thus the spectral sequence stabilizes by page $n + 1$, and by the previous proposition we obtain the following expression for E_∞. , Note for degree reasons [...] Thus [...]

Proposition 2.3. *Let X be an n-dimensional CW complex and let $\tilde{G}^*$ be a reduced generalized cohomology theory. Consider the spectral sequence, constructed above, associated to the exact couple where $D_1 = \oplus_{p,q} G^{p+q}(X^p)$ and $E_1 = \bigoplus_{p,q} G^{p+q}(X^p, X^{p-1}) \cong C^p(X; G^q(*))$. Then*

$$E_\infty = \frac{\gamma^{-1}(\alpha^n D)}{\beta(\ker \alpha^n)}.$$

We can finally prove convergence.

Proposition 2.4. *Let $p + q = k$. Given the setting of Proposition 2.3, we have*

$$E_\infty^{p,q} \cong \frac{F_{p-1}\tilde{G}^k(X)}{F_p\tilde{G}^k(X)} := \frac{\ker[\tilde{G}^k(X) \to \tilde{G}^k(X^{p-1})]}{\ker[\tilde{G}^k(X) \to \tilde{G}^n(X^p)]}.$$

Proof. We need to pick out the (p, q) position of the expression for E_∞ given in Proposition 2.3. Since (restricted to the (p, q)-position) $\ker \alpha^n = E_1^{p,q}$, we have

$$E_\infty^{p,q} = \frac{\gamma^{-1}(\operatorname{im} \alpha^n : \tilde{G}^{p+q}(X) \to \tilde{G}^{p+q}(X^p))}{\ker \gamma : \tilde{G}^{p+q}(X^p/X^{p-1}) \to \tilde{G}^{p+q}(X^p)}.$$

The exact sequence

$$0 \to \ker\gamma \to \gamma^{-1}(\operatorname{im}\alpha^n) \xrightarrow{\gamma} \operatorname{im}\alpha^n \cap \operatorname{im}\gamma \to 0$$

shows that $E_\infty^{p,q} \cong \operatorname{im}\alpha^n \cap \operatorname{im}\gamma$ restricted to the (p, q) position as above. Next, using the fact that $\operatorname{im}\gamma = \ker\alpha$, we have another exact sequence

$$0 \to \ker\alpha^n\colon \tilde{G}^{p+q}(X) \to \tilde{G}^{p+q}(X^p)) \to \ker\alpha^{n+1}\colon \tilde{G}^{p+q}(X) \to \tilde{G}^{p+q}(X^{p-1}) \to \operatorname{im}(\alpha^n)\cap\operatorname{im}\gamma \to 0.$$

Thus $E_\infty^{p,q} \cong \operatorname{im}\alpha^n \cap \operatorname{im}\gamma \cong \dfrac{F_{p-1}\tilde{G}^k(X)}{F_p\tilde{G}^k(X)}$, as desired.

□

This completes the proof of Theorem 1.4.

3 AHSS for K-theory

3.1 The simplified spectral sequence

K-theory may be the most well-known generalized cohomology theory. Here we consider complex K-theory, so that $K(X) = K^0(X)$ is the Grothendieck ring of the isomorphism classes of complex vector bundles over X. When the Atiyah-Hirzebruch spectral sequence is applied to K-theory, its statement is simplified because of Bott periodicity. We recall the version of this theorem we will use.

Theorem 3.1 (Bott periodicity)**.** *There is a natural isomorphism*

$$\tilde{K}(X) \cong \tilde{K}(\Sigma^2 X).$$

For a proof, see [7], Chapter 2.

By Bott periodicity, there are only two groups to calculate: $\tilde{K}^0(X)$ and $\tilde{K}^1(X)$. We are often interested in statements in terms of K groups, rather than reduced K groups. These are related by the equality $K^n(X) = \tilde{K}^n(X_+)$, where X_+ is X disjoint union with a point. Then one checks that $K^i(X) \cong \tilde{K}^0(X) \oplus \mathbb{Z}$ for i even and $K^i(X) \cong \tilde{K}^1(X)$ for i odd. In particular, $K^i(\mathrm{pt}) \cong \mathbb{Z}$ for i even and $K^i(\mathrm{pt}) = 0$ for i odd.

Let us now analyze what the Atiyah-Hirzebruch sequence becomes for unreduced K-theory. Compared to reduced K-theory, we must add a copy of $\mathbb{Z}$ to each row. Doing this all on column 0 of the E_2 page, we see that we can express $E_2^{p,q}$ as the unreduced cohomology $\tilde{H}^p(X; K^q(*))$. Thus the E_2 page of the spectral sequence looks like the following.

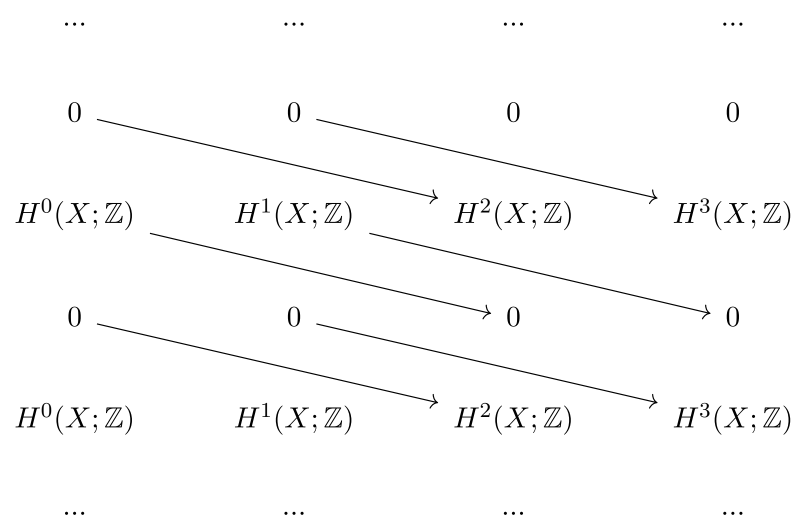

Figure 1: E_2-page

Moreover, the naturality of the isomorphism given by Bott periodicity implies that the differentials as well as the rows are periodic. Clearly all the even differentials will be 0, and we now know that the odd differentials are the same for each nonzero row. Therefore we may compress the entire spectral sequence into one row, and we obtain the following simplified statement.

Theorem 3.2 (Atiyah-Hirzebruch spectral sequence for K-theory)**.** *Let X be an n-dimensional CW-complex. Then there is a spectral sequence E_i^p with differentials $d_i^p : E_i^p \to E_i^{p+i}$ satisfying the following properties.*

- $E_2^p = H^p(X;\mathbb{Z})$.
- $E_{r+1}^p = \dfrac{\ker d_p^r}{\operatorname{im} d_r^{p-r}}$.
- $E_\infty^p = \dfrac{\ker(K^p(X) \to K^p(X^{p-2}))}{\ker(K^p(X) \to K^p(X^p))}$.

Additionally, $d_i^p = 0$ for even i.

This statement, however, can feel strange to work with because of how the pages are defined from the differentials. In fact, it may be easier to simply work with one row and remember that the differentials go to the right by an odd number of terms. Then the filtration of $K(X)$ consists of the even-indexed terms and the filtration of $K^1(X)$ consists of the odd-indexed terms. One may also prefer to use the unreduced version; the only difference is that it removes a factor of $\mathbb{Z}$ from the 0th term so that one does not have to worry about any nonzero differentials coming from that term. We will see some concrete examples in our computations in the next section.

3.2 Examples

Let us use the Atiyah-Hirzebruch spectral sequence to calculate some K-theory groups.

3.2.1 Σ_g

Let Σ_g be the real orientable surface of genus g.

Proposition 3.3. *We have*

$$K^0(\Sigma_g) \cong \mathbb{Z}^2, \qquad K^1(\Sigma_g) \cong \mathbb{Z}^{2g}.$$

Proof. Recall that the integral cohomology of Σ_g is given by $H^0(\Sigma_g, \mathbb{Z}) \cong H^2(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}$ and $H^1(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}^{2g}$. Thus the spectral sequence only has three nonzero terms: $\mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z}$. But the first nonzero differential is already of degree 3, so we see that there are in fact no nonzero differentials. Because $\mathbb{Z}$ is projective, there are no extension problems and the result follows. □

3.2.2 $\mathbb{CP}^n$

Proposition 3.4. *We have*

$$K(\mathbb{CP}^n) \cong \mathbb{Z}^{n+1}, \qquad K^1(\mathbb{CP}^n) = 0.$$

Proof. The spectral sequence begins as follows.

$$\mathbb{Z} \qquad 0 \qquad \mathbb{Z} \qquad 0 \qquad \cdots \qquad \mathbb{Z} \qquad 0$$

However, each non-zero differential d_{2k+1} is of degree $(2k+1, -2k)$ in the two-dimensional version, which in this version is simply of degree $2k+1$. Thus all differentials are 0, and because $\mathbb{Z}$ is projective there are no extension issues and we have $K(\mathbb{CP}^n) \cong \mathbb{Z}^{n+1}$ as desired. The odd terms in the spectral sequence are all 0, so $K^1(\mathbb{CP}^n) = 0$. □

Remark. We see from this computation that if a finite CW-complex X has no odd-degree cohomology, then for degree reasons the K-theory spectral sequence collapses on the second page. This implies that $K(X) \cong \oplus_k H^k(X; \mathbb{Z})$ and $K^1(X) = 0$.

In this case we can go further and actually compute $K(\mathbb{CP}^n)$ as a ring. To do this, we recall a few fundamental facts about K-theory.

Define the Chern character $\mathrm{ch} : K(X) \to H^{\mathrm{even}}(X; \mathbb{Q})$ first on vector bundles by

$$\mathrm{ch}(\xi) = \sum e^{x_i},$$

where x_i are the Chern roots of ξ. Then extend linearly; this gives a well-defined homomorphism $\mathrm{ch} : K(X) \to H^{\mathrm{even}}(X; \mathbb{Q})$.

Proposition 3.5. *After tensoring with $\mathbb{Q}$, the Chern character induces an isomorphism*

$$\mathrm{ch}_{\mathbb{Q}} \colon K(X) \otimes \mathbb{Q} \xrightarrow{\cong} \bigoplus H^{\mathrm{even}}(X; \mathbb{Q}).$$

For a proof, see [4], 38.4. We will use the following corollary.

Corollary 3.6. *The Chern character restricts to an isomorphism $\tilde{K}(S^{2n}) \xrightarrow{\cong} H^{2n}(S^{2n}; \mathbb{Z}) = \mathbb{Z} \subset \mathbb{Q} = H^{2n}(S^{2n}; \mathbb{Q})$.*

This follows from induction and Bott periodicity; see [4], 39.2 for details.

The case of $\mathbb{CP}^1$ can be calculated by hand, but is also a special case of another version of Bott periodicity.

Theorem 3.7 (Bott periodicity)**.** *There is an isomorphism of rings*

$$K(X \times S^2) \cong K(X) \otimes \mathbb{Z}[\zeta]/(\zeta - 1)^2.$$

Here, ζ is the pullback of the tautological bundle $\gamma \to \mathbb{CP}^1 \cong S^2$.

For a proof, see [7], Ch. 2. We will now calculate the ring structure of $K(\mathbb{CP}^n)$.

Proposition 3.8. *Let $\gamma = \zeta - 1$, where ζ is the line bundle over $\mathbb{CP}^n$ obtained by pullback of the tautological bundle. Then $K(\mathbb{CP}^n) \cong \mathbb{Z}[\gamma]/\gamma^{n+1}$ as rings.*

Proof. Let $x \in H^2(\mathbb{CP}^n;\mathbb{Z})$ be the generator corresponding to the Chern root of ζ. Then we have

$$\operatorname{ch}\gamma = \operatorname{ch}\zeta - 1 = e^x - 1 = x + \frac{x^2}{2} + \cdots$$

and thus

$$\operatorname{ch}\gamma^k = x^k + \frac{k}{2}x^{k+1} + \cdots .$$

Furthermore, since $x^{n+1} = 0$, we have $\gamma^{n+1} = 0$. This computation shows that $1, \gamma, \ldots, \gamma^n$ are linearly independent in $K(\mathbb{CP}^n)$. By Proposition 3.4, this means they generate $K(\mathbb{CP}^n)$ over $\mathbb{Q}$. To show they generate it over $\mathbb{Z}$, we use induction. It is true for $\mathbb{CP}^1$, so assume it is true for $\mathbb{CP}^{n-1}$. Then for any

$$\alpha = a_0 + a_1\gamma + \cdots + a_n\gamma^n \in K(\mathbb{CP}^n),$$

we have that its restriction to $\mathbb{CP}^{n-1}$, which is just $a_0 + a_1\gamma + \cdots + a_{n-1}\gamma^{n-1}$, must be in $\mathbb{Z}[\gamma]$. It remains to show $r_n \in \mathbb{Z}$. Consider the portion of the long exact sequence

$$\cdots \to K(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \to K(\mathbb{CP}^n) \to K(\mathbb{CP}^{n-1}) \to \cdots .$$

Because $r_n\gamma^n \in K(\mathbb{CP}^n)$ gets sent to $0 \in K(\mathbb{CP}^{n-1})$, it is the image of some element of $K(\mathbb{CP}^n, \mathbb{CP}^{n-1}) = \tilde{K}(S^{2n})$. By Corollary 3.6, such an element has an integral Chern class, so $r_n \in \mathbb{Z}$ as desired. □

3.2.3 $\mathbb{RP}^n$

Recall that the reduced real cohomology of $\mathbb{RP}^n$ is 0 in odd degrees (and 0) and $\mathbb{Z}/2\mathbb{Z}$ in positive even degrees. Therefore, the reduced version of the spectral sequence begins as follows.

$$0 \qquad 0 \qquad \mathbb{Z}/2\mathbb{Z} \qquad 0 \qquad \cdots \qquad 0 \qquad \mathbb{Z}/2\mathbb{Z} \qquad (\text{if } n \text{ odd})\ \mathbb{Z}$$

Again we have that all differentials are trivial because there is essentially no odd cohomology. Thus we have that

$$K(\mathbb{RP}^n) \cong \mathbb{Z} \oplus \tilde{K}(\mathbb{RP}^n) = \mathbb{Z} \oplus G$$

where G is a group of order $2^{\lfloor n/2 \rfloor}$, and $K^1(\mathbb{RP}^n) = 0$ for even n and $\mathbb{Z}$ for odd n. The spectral sequence alone will not determine G. Moreover, the ring structure has not been elucidated. However, we at least know the group structure.

Proposition 3.9. *We have*

$$K(\mathbb{RP}^n) \cong \mathbb{Z} \oplus \mathbb{Z}/2^{\lfloor n/2 \rfloor}\mathbb{Z}.$$

The torsion part can be calculated through analyzing the embedding $\mathbb{RP}^{2m} \hookrightarrow \mathbb{CP}^{2m}$ and using the calculation of $K(\mathbb{CP}^n)$. For complete details, see [4], 39.3.

Remark. The fact that $K(\mathbb{RP}^n) \cong \mathbb{Z} \oplus \mathbb{Z}/2^{\lfloor n/2 \rfloor}\mathbb{Z}$ means that its torsion part is different from that of $\bigoplus H^{\text{even}}(\mathbb{RP}^n;\mathbb{Z})$, though they have the same order.

3.2.4 $\mathbb{RP}^2 \times \mathbb{RP}^4$

Thus far, all the differentials have been 0 in the examples we have considered. When $X = \mathbb{RP}^2 \times \mathbb{RP}^4$, this is not the case. In fact, through the calculations done in this section we will be able to determine the differential d_3. To begin, we calculate the K-theory of this space using a Künneth formula for K-theory, proven by Atiyah in [3].

Theorem 3.10. *Let X and Y be finite CW-complexes. There is a natural $\mathbb{Z}/2\mathbb{Z}$-graded exact sequence:*

$$0 \to K^*(X) \otimes K^*(Y) \xrightarrow{\alpha} K^*(X \times Y) \xrightarrow{\beta} \mathrm{Tor}(K^*(X), K^*(Y)) \to 0.$$

In this grading, $\deg \alpha = 0, \deg \beta = 1.$

Here $K^*(X) = K^0(X) \oplus K^1(X)$. The grading implies there are two exact sequence, where one of the two 'factors' of $K^*(X \times Y)$ has even total degree and the other has odd total degree.

Remark. As Atiyah mentions, this is not a general formula for generalized cohomology theories. For instance, it fails for real K-theory.

Proposition 3.11. *We have*

$$K^0(\mathbb{RP}^2 \times \mathbb{RP}^4) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}/4\mathbb{Z}, \qquad K^1(\mathbb{RP}^2 \times \mathbb{RP}^4) \cong \mathbb{Z}/2\mathbb{Z}.$$

Proof. This is a direct application of Proposition 3.9 and Atiyah's Künneth formula. We have $K^0(\mathbb{RP}^2) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}), K^1(\mathbb{RP}^2) = 0, K^0(\mathbb{RP}^4) = \mathbb{Z} \oplus \mathbb{Z}/4, K^1(\mathbb{RP}^4) = 0$. Indeed, we have the following exact sequences.

$$\begin{aligned} 0 \to K^0(\mathbb{RP}^2) \otimes K^0(\mathbb{RP}^4) \oplus K^1(\mathbb{RP}^2) \otimes K^1(\mathbb{RP}^4) \to K^0(\mathbb{RP}^2 \times \mathbb{RP}^4) \to \\ \mathrm{Tor}(K^0(\mathbb{RP}^2), K^1(\mathbb{RP}^4)) \oplus \mathrm{Tor}(K^1(\mathbb{RP}^2), K^0(\mathbb{RP}^4)) \to 0 \end{aligned}$$

so

$$0 \to \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}/4\mathbb{Z} \to 0 \to 0$$

and

$$\begin{aligned} 0 \to K^0(\mathbb{RP}^2) \otimes K^1(\mathbb{RP}^4) \oplus K^1(\mathbb{RP}^2) \otimes K^0(\mathbb{RP}^4) \to K^1(\mathbb{RP}^2 \times \mathbb{RP}^4) \to \\ \mathrm{Tor}(K^0(\mathbb{RP}^2), K^0(\mathbb{RP}^4)) \oplus \mathrm{Tor}(K^1(\mathbb{RP}^2), K^1(\mathbb{RP}^4)) \to 0 \end{aligned}$$

so

$$0 \to 0 \to K^0(\mathbb{RP}^4) \to \mathbb{Z}/2\mathbb{Z} \to 0$$

The result follows. □

Now let us see what happens when we use the Atiyah-Hirzebruch spectral sequence. By the Künneth formula, we have

$$H^i(\mathbb{RP}^2 \times \mathbb{RP}^4; \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ (\mathbb{Z}/2)^2 & i = 2 \\ \mathbb{Z}/2 & i = 3 \\ (\mathbb{Z}/2)^2 & i = 4 \\ \mathbb{Z}/2 & i = 5 \\ \mathbb{Z}/2 & i = 6. \end{cases}$$

Consider the Atiyah-Hirzebruch sequence for reduced K-theory; we will draw what the E_3 page looks like.

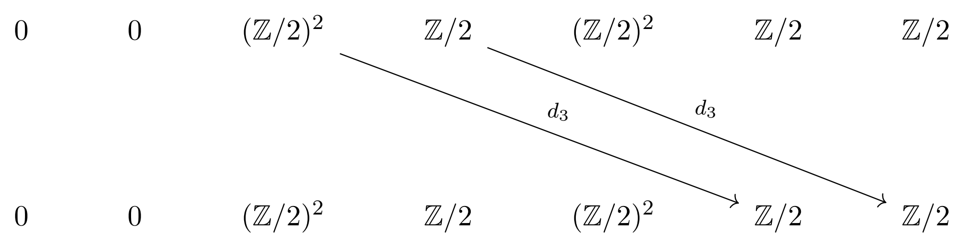

Figure 2: E_3-page

All higher differentials vanish for degree reasons. Thus we know that the even terms must give a filtration of $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/4$. This implies that one of the two d_3 arrows drawn is 0, and the other is surjective. This also agrees with our calculation that $K^1(\mathbb{RP}^2 \times \mathbb{RP}^4) = \mathbb{Z}/2$. We have thus shown that the d_3 differential is not always 0.

In fact, it is not hard to go further and show that d_3 is a stable cohomology operation. Stable cohomology operations of degree 3 are classified by

$$H^{n+3}(K(\mathbb{Z}, n); Z) = \mathbb{Z}/2 \quad (n \geq 3).$$

Thus there is a unique nonzero one, which is given by the composition

$$H^n(X;\mathbb{Z}) \xrightarrow{\rho_2} H^n(X;\mathbb{Z}/2) \xrightarrow{\mathrm{Sq}^2} H^{n+2}(X;\mathbb{Z}/2) \xrightarrow{\beta} H^{n+3}(X;\mathbb{Z}),$$

where ρ is reduction modulo 2 and β is the Bockstein homomorphism[2]. Let us denote it by $\tilde{\mathrm{Sq}}^3$. We then have the following proposition.

Proposition 3.12. *The differential d_3 of the Atiyah-Hirzebruch spectral sequence is given by*

$$d_3 = \tilde{\mathrm{Sq}}^3 : H^n(X;\mathbb{Z}) \to H^{n+3}(X;\mathbb{Z}).$$

In [4], 39.3, this is proven by exhibiting a different space (from $\mathbb{RP}^2 \times \mathbb{RP}^4$) where d_3 does not vanish.

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