Five Problems in Honor of Doron Zeilberger

Compiled by Caleb

1 Deep theorems[1]

1. Let the n^{th} Motzkin number be the number of ways to draw any number of non-intersecting chords joining n labeled points on a circle. The first few values are given by $M_0 = M_1 = 1, M_2 = 2, M_3 = 4, M_4 = 9, M_5 = 21, M_6 = 51$. This is entry A001006 in the OEIS, in which several nice combinatorial interpretations are given.

   Prove that for any prime p, we have

$$\Big(\sum_{i=0}^{10p-1} M_i\Big)\Big(\sum_{i=0}^{100p-1} M_i\Big) \equiv \pm 2462596841775734241642208105453427790433576916592966 \pmod{p}.$$

2. A partition λ of n is a sequence of nonincreasing positive integers that sum to n. A partition can be presented as a Young tableau, and the hook length of each cell is the number of squares to the right or below it, including itself. See the figure below for an example.

7	4	3	1
5	2	1	
2			
1			

Figure 1: Hook lengths for $\lambda = (4, 3, 1, 1) \vdash 9$

   A partition is called (s,t)-core if it has no hook-length of s or t. It is an interesting problem to enumerate the number of (s,t)-core partitions of any size. For instance, Jaclyn Anderson proved that if $(s,t) = 1$, then there are exactly $\frac{(s+t-1)!}{s!t!}$ (s,t)-core partitions. Also, it was proven by Yan et al. that the number of $(2n+1, 2n+3)$-core partitions with distinct parts is 4^n. This last elegant result motivates the following equally elegant theorem:

   Show that the fourth moment (about the mean) of the random variable 'size' defined on $(2n+1, 2n+3)$-core partitions with distinct parts is

$$\frac{1}{54499737600} \cdot (1743712560n^{12} + 13490284234n^{11} + 45408125279n^{10} + 87568584895n^{9} + 109173019890n^{8} + 97494786972n^{7} + 68082466947n^{6} + 34594762895n^{5} + 8734303600n^{4} + 3269131844n^{3} + 7648567524n^{2} + 4135638960n).$$

3. Morley's theorem is a beautiful result in classical Euclidean geometry, stating that the triangle formed by the angle trisectors of any triangle is equilateral. John Conway gave a proof with elementary geometry, and Alain Connes gave an algebraic proof extending to general fields. DZ joins this pantheon of great mathematicians who contributed to our understanding of this phenomenon with the following marvelous extension:

[1] Or as DZ would call them, Deep Incestuous Relationships between Overdetermined Inbred Mathematical Objects: http://sites.math.rutgers.edu/~zeilberg/Opinion44.html

In any triangle ABC, draw the rays that divide each angle into four equal parts. Take the three intersections given by the rays closest to each edge; call these points D, E, and F so that A is opposite D, B is opposite E, and C is opposite F. Let $A12 = \frac{DE^2}{BC}, A23 = \dfrac{EF^2}{BC}, A31 = \dfrac{FD^2}{BC}$. Then show that

$-8*A12^14+88*A12^13*A23+32*A12^13*A31-516*A12^12*A23^2-200*A12^12*A23*A31-88*A12^12*A31^2+$
$2016*A12^11*A23^3+512*A12^11*A23^2*A31+464*A12^11*A23*A31^2+192*A12^11*A31^3-5714*A12^10*A23^4+$
$112*A12^10*A23^3*A31-1704*A12^10*A23^2*A31^2-688*A12^10*A23*A31^3-328*A12^10*A31^4+12230*A12^9*$
$A23^5-5184*A12^9*A23^4*A31+5552*A12^9*A23^3*A31^2+1344*A12^9*A23^2*A31^3+776*A12^9*A23*A31^4+$
$480*A12^9*A31^5-20113*A12^8*A23^6+19294*A12^8*A23^5*A31-17218*A12^8*A23^4*A31^2+1440*A12^8*A23^3*$
$A31^3-2172*A12^8*A23^2*A31^4-664*A12^8*A23*A31^5-600*A12^8*A31^6+25584*A12^7*A23^7-41732*A12^7*$
$A23^6*A31+43656*A12^7*A23^5*A31^2-16736*A12^7*A23^4*A31^3+6272*A12^7*A23^3*A31^4+1472*A12^7*A23^2*$
$A31^5+224*A12^7*A23*A31^6+640*A12^7*A31^7-25192*A12^6*A23^8+61552*A12^6*A23^7*A31-82472*A12^6*$
$A23^6*A31^2+55736*A12^6*A23^5*A31^3-25132*A12^6*A23^4*A31^4+4064*A12^6*A23^3*A31^5-1968*A12^6*$
$A23^2*A31^6+224*A12^6*A23*A31^7-600*A12^6*A31^8+19108*A12^5*A23^9-64940*A12^5*A23^8*A31+112656*$
$A12^5*A23^7*A31^2-110348*A12^5*A23^6*A31^3+68380*A12^5*A23^5*A31^4-23104*A12^5*A23^4*A31^5+4064*$
$A12^5*A23^3*A31^6+1472*A12^5*A23^2*A31^7-664*A12^5*A23*A31^8+480*A12^5*A31^9-11018*A12^4*A23^10+$
$49596*A12^4*A23^9*A31-110136*A12^4*A23^8*A31^2+145168*A12^4*A23^7*A31^3-123950*A12^4*A23^6*A31^4+$
$68380*A12^4*A23^5*A31^5-25132*A12^4*A23^4*A31^6+6272*A12^4*A23^3*A31^7-2172*A12^4*A23^2*A31^8+$
$776*A12^4*A23*A31^9-328*A12^4*A31^10+4704*A12^3*A23^11-27100*A12^3*A23^10*A31+75784*A12^3*$
$A23^9*A31^2-128920*A12^3*A23^8*A31^3+145168*A12^3*A23^7*A31^4-110348*A12^3*A23^6*A31^5+55736*$
$A12^3*A23^5*A31^6-16736*A12^3*A23^4*A31^7+1440*A12^3*A23^3*A31^8+1344*A12^3*A23^2*A31^9-688*A12^3*$
$A23*A31^10+192*A12^3*A31^11-1414*A12^2*A23^12+10176*A12^2*A23^11*A31-35320*A12^2*A23^10*A31^2+$
$75784*A12^2*A23^9*A31^3-110136*A12^2*A23^8*A31^4+112656*A12^2*A23^7*A31^5-82472*A12^2*A23^6*A31^6+$
$43656*A12^2*A23^5*A31^7-17218*A12^2*A23^4*A31^8+5552*A12^2*A23^3*A31^9-1704*A12^2*A23^2*A31^10+$
$464*A12^2*A23*A31^11-88*A12^2*A31^12+270*A12*A23^13-2388*A12*A23^12*A31+10176*A12*A23^11*$
$A31^2-27100*A12*A23^10*A31^3+49596*A12*A23^9*A31^4-64940*A12*A23^8*A31^5+61552*A12*A23^7*$
$A31^6-41732*A12*A23^6*A31^7+19294*A12*A23^5*A31^8-5184*A12*A23^4*A31^9+112*A12*A23^3*A31^10+$
$512*A12*A23^2*A31^11-200*A12*A23*A31^12+32*A12*A31^13-25*A23^14+270*A23^13*A31-1414*A23^12*$
$A31^2+4704*A23^11*A31^3-11018*A23^10*A31^4+19108*A23^9*A31^5-25192*A23^8*A31^6+25584*A23^7*A31^7-$
$20113*A23^6*A31^8+12230*A23^5*A31^9-5714*A23^4*A31^10+2016*A23^3*A31^11-516*A23^2*A31^12+88*A23*$
$A31^13-8*A31^14+288*A12^13-2316*A12^12*A23-1596*A12^12*A31+10344*A12^11*A23^2+8688*A12^11*$
$A23*A31+4572*A12^11*A31^2-32278*A12^10*A23^3-25926*A12^10*A23^2*A31-16152*A12^10*A23*A31^2-$
$9220*A12^10*A31^3+76434*A12^9*A23^4+46466*A12^9*A23^3*A31+36766*A12^9*A23^2*A31^2+21104*A12^9*$
$A23*A31^3+12708*A12^9*A31^4-141044*A12^8*A23^5-39170*A12^8*A23^4*A31-61454*A12^8*A23^3*A31^2-$
$45028*A12^8*A23^2*A31^3-10932*A12^8*A23*A31^4-11656*A12^8*A31^5+202428*A12^7*A23^6-38328*A12^7*$
$A23^5*A31+108832*A12^7*A23^4*A31^2+59096*A12^7*A23^3*A31^3+17244*A12^7*A23^2*A31^4-2144*A12^7*$
$A23*A31^5+4904*A12^7*A31^6-224024*A12^6*A23^7+183892*A12^6*A23^6*A31-223264*A12^6*A23^5*A31^2-$
$20584*A12^6*A23^4*A31^3-5852*A12^6*A23^3*A31^4-16952*A12^6*A23^2*A31^5+11696*A12^6*A23*A31^6+$
$4904*A12^6*A31^7+189360*A12^5*A23^8-309268*A12^5*A23^7*A31+394056*A12^5*A23^6*A31^2-135624*A12^5*$
$A23^5*A31^3+31864*A12^5*A23^4*A31^4+34636*A12^5*A23^3*A31^5-16952*A12^5*A23^2*A31^6-2144*A12^5*$
$A23*A31^7-11656*A12^5*A31^8-120560*A12^4*A23^9+318912*A12^4*A23^8*A31-514504*A12^4*A23^7*A31^2+$
$401896*A12^4*A23^6*A31^3-206552*A12^4*A23^5*A31^4+31864*A12^4*A23^4*A31^5-5852*A12^4*A23^3*A31^6+$
$17244*A12^4*A23^2*A31^7-10932*A12^4*A23*A31^8+12708*A12^4*A31^9+56412*A12^3*A23^10-218600*A12^3*$
$A23^9*A31+459916*A12^3*A23^8*A31^2-546072*A12^3*A23^7*A31^3+401896*A12^3*A23^6*A31^4-135624*$
$A12^3*A23^5*A31^5-20584*A12^3*A23^4*A31^6+59096*A12^3*A23^3*A31^7-45028*A12^3*A23^2*A31^8+21104*$
$A12^3*A23*A31^9-9220*A12^3*A31^10-18514*A12^2*A23^11+99410*A12^2*A23^10*A31-274400*A12^2*A23^9*$
$A31^2+459916*A12^2*A23^8*A31^3-514504*A12^2*A23^7*A31^4+394056*A12^2*A23^6*A31^5-223264*A12^2*$
$A23^5*A31^6+108832*A12^2*A23^4*A31^7-61454*A12^2*A23^3*A31^8+36766*A12^2*A23^2*A31^9-16152*A12^2*$
$A23*A31^10+4572*A12^2*A31^11+3870*A12*A23^12-27838*A12*A23^11*A31+99410*A12*A23^10*A31^2-$
$218600*A12*A23^9*A31^3+318912*A12*A23^8*A31^4-309268*A12*A23^7*A31^5+183892*A12*A23^6*A31^6-$
$38328*A12*A23^5*A31^7-39170*A12*A23^4*A31^8+46466*A12*A23^3*A31^9-25926*A12*A23^2*A31^10+8688*$
$A12*A23*A31^11-1596*A12*A31^12-400*A23^13+3870*A23^12*A31-18514*A23^11*A31^2+56412*A23^10*$
$A31^3-120560*A23^9*A31^4+189360*A23^8*A31^5-224024*A23^7*A31^6+202428*A23^6*A31^7-141044*A23^5*$
$A31^8+76434*A23^4*A31^9-32278*A23^3*A31^10+10344*A23^2*A31^11-2316*A23*A31^12+288*A31^13-4050*$
$A12^12+26460*A12^11*A23+27270*A12^11*A31-90918*A12^10*A23^2-121842*A12^10*A23*A31-103257*$
$A12^10*A31^2+208020*A12^9*A23^3+291276*A12^9*A23^2*A31+326826*A12^9*A23*A31^2+262038*A12^9*$

$A31^3-353360*A12^8*A23^4-427172*A12^8*A23^3*A31-641694*A12^8*A23^2*A31^2-526040*A12^8*A23*A31^3-$
$493082*A12^8*A31^4+468104*A12^7*A23^5+409052*A12^7*A23^4*A31+796476*A12^7*A23^3*A31^2+833240*$
$A12^7*A23^2*A31^3+542876*A12^7*A23*A31^4+709668*A12^7*A31^5-496388*A12^6*A23^6-190420*A12^6*A23^5*$
$A31-845150*A12^6*A23^4*A31^2-571124*A12^6*A23^3*A31^3-945628*A12^6*A23^2*A31^4-225752*A12^6*$
$A23*A31^5-801270*A12^6*A31^6+423224*A12^5*A23^7-94736*A12^5*A23^6*A31+747072*A12^5*A23^5*A31^2+$
$333708*A12^5*A23^4*A31^3+366664*A12^5*A23^3*A31^4+874488*A12^5*A23^2*A31^5-225752*A12^5*A23*A31^6+$
$709668*A12^5*A31^7-288346*A12^4*A23^8+286576*A12^4*A23^7*A31-704988*A12^4*A23^6*A31^2+269308*$
$A12^4*A23^5*A31^3-726660*A12^4*A23^4*A31^4+366664*A12^4*A23^3*A31^5-945628*A12^4*A23^2*A31^6+$
$542876*A12^4*A23*A31^7-493082*A12^4*A31^8+153180*A12^3*A23^9-291490*A12^3*A23^8*A31+565940*A12^3*$
$A23^7*A31^2-406424*A12^3*A23^6*A31^3+269308*A12^3*A23^5*A31^4+333708*A12^3*A23^4*A31^5-571124*$
$A12^3*A23^3*A31^6+833240*A12^3*A23^2*A31^7-526040*A12^3*A23*A31^8+262038*A12^3*A31^9-60950*A12^2*$
$A23^10+186630*A12^2*A23^9*A31-412377*A12^2*A23^8*A31^2+565940*A12^2*A23^7*A31^3-704988*A12^2*$
$A23^6*A31^4+747072*A12^2*A23^5*A31^5-845150*A12^2*A23^4*A31^6+796476*A12^2*A23^3*A31^7-641694*$
$A12^2*A23^2*A31^8+326826*A12^2*A23*A31^9-103257*A12^2*A31^10+16500*A12*A23^11-72700*A12*A23^10*$
$A31+186630*A12*A23^9*A31^2-291490*A12*A23^8*A31^3+286576*A12*A23^7*A31^4-94736*A12*A23^6*$
$A31^5-190420*A12*A23^5*A31^6+409052*A12*A23^4*A31^7-427172*A12*A23^3*A31^8+291276*A12*A23^2*$
$A31^9-121842*A12*A23*A31^10+27270*A12*A31^11-2500*A23^12+16500*A23^11*A31-60950*A23^10*$
$A31^2+153180*A23^9*A31^3-288346*A23^8*A31^4+423224*A23^7*A31^5-496388*A23^6*A31^6+468104*A23^5*$
$A31^7-353360*A23^4*A31^8+208020*A23^3*A31^9-90918*A23^2*A31^10+26460*A23*A31^11-4050*A31^12=0.$

4. Let $p \equiv 1 \pmod 3$ be a prime. Prove that

$$2^{\frac{p-1}{3}} \equiv 2^{-\frac{1}{3}} \pmod p.$$

2 Open problems

5. It seems that the set $\{2^{-\frac{1}{3}} \pmod p\}$ is not finite. Can you prove or disprove this assertion?

Solutions

1. http://sites.math.rutgers.edu/~zeilberg/Opinion147.html
2. http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/arminV2.pdf
3. http://sites.math.rutgers.edu/~zeilberg/tokhniot/oGeneralizedMorley1
4. http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/ctcong.pdf
5. http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/ctcong.pdf