INTRO TO THE GAUSS-MANIN CONNECTION

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In our study of the hypergeometric differential equation, we saw how loops around the singular points transformed solutions in a certain way. This illustrated the Riemann-Hilbert correspondence, which gives a correspondence between linear differential equations over some space S with singularities at some prescribed points $\{P_i\}$ and representations of the fundamental group $\pi_1(S\backslash\{P_i\})$. Today we are going to relate these ideas to a third mathematical object: connections on vector bundles. These gives a new perspective on what derivatives, and thus differential equations, really are. We will discuss how all these objects sometimes arise from the cohomology of algebraic varieties; i.e., are 'motivic.'

1. Elliptic curves

1.1. **Topological classification of surfaces.** Compact connected orientable surfaces are topologically classified by their genus: how many holes they have. As it turns out, elliptic curves have genus 1, and are thus tori. It will be useful for us to identify tori with $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. That is, one takes the complex plane and mod out by an integral lattice within it, so every point is identified with one inside the parallelogram defining the lattice.

One can see this topologically by taking a piece of paper and folding both pairs of edges together to form a donut. In fact, one can do this for higher genus, though it is more difficult to visualize. Note that we can construct a surface of genus 2 by taking to tori, cutting open a disk in each of them, and gluing them along the disc. If we do this schematically on the sheets of paper which represent the tori, we end up with an octagon with certain sides identified. We can do this repeatedly to add more holes. The general result is that one can construct a surface of genus g by taking a $4g$-gon and appropriately identifying its sides.

This is the topological classification of surfaces. This means that every (compact connected orientable) surface can be deformed into one of these; i.e., there exists a homeomorphism between any two surfaces of the same genus. However, these maps may not be algebraic or analytic. This means that there may be multiple surfaces with a fixed genus which are not isomorphic algebraically (i.e. via a polynomial) or analytically (i.e. via a holomorphic map). In fact, the study of the algebraic isomorphism classes of surfaces with a fixed genus is one of the most interesting things in all of mathematics.

1.2. **Weierstrass equations of elliptic curves.** Elliptic curves over the complex numbers are *curves*, so they are one-dimensional over $\mathbb{C}$, so they are two-dimensional over $\mathbb{R}$ and are thus topological surfaces. The (affine) equation defining an elliptic curve can be given the form

$$y^2 = x^3 + ax + b,$$

the Weierstrass equation of the elliptic curve. The question now is: how does this make it into a surface of genus 1? As a matter of fact, one needs to add a point at infinity to make this work; rigorously, one adds another variable z and considers the solutions to $y^2z = x^3 + axz^2 + bz^3$ in projective space, but we will not discuss this in detail here.

The argument we now explain is inspired by Riemann's original conception of Riemann surfaces. One wants to trace out the surface by using y as a local coordinate. If we consider the equation

$$y = \sqrt{x^3 + ax + b},$$

then really y should take both possible values of the square root on the elliptic curve, and by going around the surface one ought to be able to pass through the different values. How to make this more precise? First, let us assume we can factorize $x^3 + ax + b = x(x-1)(x-\lambda)$ for some complex number $\lambda \neq 0, 1$ (elliptic curves can indeed always be put into this form). Then if we go around 0, 1, λ, or ∞, we should end up on a different copy of the sphere, but if we go around two at a time we should end up on the same sphere. A way to visualize this is to take two copies of the Riemann sphere with cuts from 0 to 1 and λ to ∞ on both of them and glue them together along the cuts. This ends up producing a torus.

To write down an isomorphism between the elliptic curve and a torus, we begin with any basepoint P on the torus and consider the holomorphic differential form $\omega = \frac{dx}{y}$. For any point Q on the elliptic curve, we assign the complex number $\int_P^Q \omega$. This is almost well-defined. Indeed, in complex analysis we learn that the integral of any holomorphic differential form along a closed contour in a simply connected region is 0, which means that any two paths we take will give us the same integral provided there are no holes inside. Well, there are two ways to go around holes in a torus, given by the two cycles around the torus, which we will call γ_1 and γ_2. Of course one can take multiples of these too, so if $\int_{\gamma_1} \omega = \omega_1$ and $\int_{\gamma_2} \omega = \omega_2$, the integral is defined up to an integer combination of ω_1 and ω_2. One can normalize the elliptic curve by scaling to assume one of these periods is 1, and the other lies in the upper half plane. Then this gives the desired isomorphism of E with $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, with τ in the upper half plane.

1.3. **Moduli of elliptic curves.** The term 'moduli space' refers to a space in which every point corresponds to a geometric object. The different variations of the geometric object are put together in a natural way in a moduli space; this can be made very precise mathematically, though we do not have the space to do this here. But we have just seen how to parameterize elliptic curves. The parameter τ determines the isomorphism class of the elliptic curve, though if a different τ gives the same integral lattice $\langle 1, \tau \rangle = \langle 1, \tau' \rangle$, then they represent the same elliptic curve. Since the automorphisms of an integral lattice are given by the action of $\mathrm{SL}(2, \mathbb{Z})$, we see that the moduli space of elliptic curves is given by the quotient of the upper half plane with $\mathrm{SL}(2, \mathbb{Z})$.

However, one can also ask how the elliptic curve varies as we vary the parameter λ. Thinking of the set of elliptic curves as depending on the parameter λ is known as the Legendre family of elliptic curves. As λ varies, so does the period lattice consisting of the integrals of ω along the cycles. It turns out these periods satisfy a differential equation depending on λ known as the Picard-Fuchs differential equation, which we will discuss next.

2. The Picard-Fuchs differential equation

2.1. **de Rham cohomology classes.** We will not have the space here to develop either the theory of cohomology or the theory of differential forms, which is necessary to understand what is happening here. The idea though is that cohomology classes can be considered as differential forms, which are in some sense dual to cycles. The pairing between them is given by integration. In the situation of elliptic curves, just as the homology of an elliptic curve is a 2-dimensional vector space (basis given by γ_1, γ_2), the cohomology $H^1_{dR}(E)$ also has dimension 2. One of the basis elements can be taken to be $\omega = \frac{dx}{y}$, and we can take the other to be its derivative ω' with respect to λ. (This form is not holomorphic but is a meromorphic form with vanishing residue, and thus defines an element of de Rham cohomology.) That means that further derivatives must be linearly dependent with ω and ω'. Writing them out, we have

$$\omega = \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}, \quad \omega' = \frac{dx}{2\sqrt{x(x-1)(x-\lambda)^3}}, \quad \omega'' = \frac{3dx}{4\sqrt{x(x-1)(x-\lambda)^5}}.$$

To find a linear relation among them, we use the fact that the exterior derivative of the meromorphic function $\frac{2y}{(x-\lambda)^2}$ is 0, and find a linear combination of $\omega, \omega', \omega''$ that equals it. We end up with the Picard-Fuchs differential equation:

$$\lambda(\lambda-1)\omega'' + (2\lambda-1)\omega' + \frac{1}{4}\omega = 0.$$

If one wants to express this in terms of periods, by integrating both of the periods satisfy the same differential equation with variable λ as ω above. Note that this is a hypergeometric equation; we discussed the power series solutions to it found by Euler. This gives an explicit way to compute periods of elliptic curves.

2.2. **The cohomology of varieties.** This phenomenon of differential equations arising from a moduli space is a common occurence. The cohomology of a variety $H^*(X)$ is a collection of vector spaces that describes its topology. We have seen how it can be measured by differential forms, and is dual to the cycles that can be found on the variety. In general, if we are given a morphism $f\colon X \to S$, we can think of the base as parameterizing many different varieties. That is, we can associate to each point $s \in S$ the fiber $X_s = f^{-1}(s)$. We can then consider how the cohomology of the fibers vary. It turns out that in nice scenarios, they form a vector bundle. Loosely speaking, this means the total space of all the cohomology is locally of the form $U \times \mathbb{C}^n$. This was in the case of elliptic curves, where the morphism is given by the projection of the Legendre family to $\mathbb{P}^1 - \{0, 1, \infty\}$.

However, a vector bundle itself does not itself give a local system of vector spaces, or the associated differential equation. There must be a way to connect the different fibers $\mathbb{C}^n$ which represent the cohomology at each point. This is given by a connection. These connections give a way to differentiate sections, and those whose derivative 0 are the flat sections, and they give the local system or the solutions to the associated differential equation.

3. The Gauss-Manin connection

3.1. **Connections on vector bundles.** The simplest complex vector bundle over a space X is the trivial one, given by $X \times \mathbb{C}^n$. In general, a vector bundle is a space $E \to X$ which locally looks like that. In particular, the fiber over every point is given by $\mathbb{C}^n$, but the whole space might be twisted, like the Möbius strip (a real vector bundle).

In the case of a trivial vector bundle, there is a canonical way to identify the fibers over different points. To do this for a general vector bundle, we need the notion of a connection. This gives a new way to think about derivatives. A key property of derivatives is the fact that they satisfy the Leibniz (or product) rule.

Connections generalize directional derivatives. The goal is to use a vector field to act on sections of a vector bundle. For example, a continuous function can be thought of as a section of the trivial 1-dimensional vector bundle, and one can differentiate it at any point using a vector field by applying the directional derivative to it along the tangent vector given by the vector field at the chosen point. The flat sections are those which become 0 when applying the connection; in the simplest case described above this would just be when the derivative of a function is 0, in which case we get that the flat functions are just the constant functions (assuming X is connected).

Now we can give the formal definition.

Definition 3.1. *Let $E \to X$ be a vector bundle, let $s \in \Gamma(E)$ be a section of E and let $\xi \in \mathfrak{X}(X)$ be a vector field on X. Then a connection ∇ is a map $\nabla\colon \mathfrak{X}(X) \times \Gamma(E) \to \Gamma(E)$ such that*

- *$\nabla_\xi(s)$ is $\mathbb{C}$-linear in s and C^∞-linear in ξ.*
- *For f a smooth function, we have*

$$\nabla_\xi(fs) = f\nabla_\xi(s) + \xi(f) \cdot s.$$

Here, $\xi(f)$ refers to the action of ξ on f, which as we discussed above is simply the directional derivative.

It is often more convenient to write a connection in a different form, with $D\colon E \to E \otimes \Omega^1_X$ (where now we think of E as a locally free sheaf) satisfying $\nabla(fs) = f\nabla(s) + s \otimes df$. We can now make explicit the relation to differential equations. If we have local coordinates $(x_1, \dots, x_n)$ for our space, then locally a section of a rank m vector bundle is given by an m-tuple $(s_1, \dots, s_m)$ of functions. Then there is a canonical connection just given by the exterior derivative d, and the difference $\nabla - d$ must be a matrix of one-forms multiplied by s. Explicitly, if e_i is the ith coordinate function and $\nabla(e_i) = \sum_{j=1}^m A_i^j e_j$, then since $s = \sum_{i=1}^m s_i e_i$, the Leibniz rule gives

$$\nabla(s) = ds + As,$$

where A is the matrix of one-forms given by A_i^j. The flat sections are those for which $\nabla(s) = 0$, and we see that these correspond precisely to the functions which satisfy the linear differential equation $\mathbf{x}' + A\mathbf{x} = 0$.

3.2. **Definition of the Gauss-Manin connection.** Let $f\colon X \to S$ be a smooth proper morphism. For any integer p, there is a vector bundle on S, defined by $R^p f_* \Omega^\bullet_{X/S}$, whose fibers at s give the pth de Rham cohomology groups of X_s. We wish to define a connection on this vector bundle, which will end up coinciding with the one we worked out explicitly in the example of the Legendre family of elliptic curves. To do this, we note that the complex $\Omega^\bullet_{X/S}$ is a resolution of $f^{-1}\mathcal{O}_S$. This implies that we can write

$$R^p f_* \Omega^\bullet_{X/S} = R^p f_* \mathbb{C}_X \otimes \mathcal{O}_S.$$

The latter expression has a connection simply given by $\nabla(s \otimes f) = s \otimes df$. This is the Gauss-Manin connection.

It turns out that there is an alternate way to define the Gauss-Manin connection, discovered by Katz-Oda, that works for all smooth morphisms. This allows for a deeper analysis of the connection. In general, it is a deep and interesting question ask when local systems come from the cohomology of algebraic varieties, such as in the Gauss-Manin connection. Such situations are called ‘motivic,’ in reference to the deep idea of motives in algebraic geometry coming from Grothendieck. Katz and Simpson have made interesting progress and conjectures on this question.