Stable homotopy theory

Notes by Caleb Ji

These are my personal notes for a course taught (over Zoom) by Paul ValKoughnett in 2021. The website for the course is here and an outline of the course is here. The recordings of the class are posted here. Sometimes I will try to fill out details I missed during lecture, but sometimes I won't have time and thus some notes may be incomplete or even missing. I take all responsibility for the errors in these notes, and I welcome any emails at caleb.ji@columbia.edu with any corrections!

Contents

1 Overview of spectra

Lecturer: Paul VanKoughnett, Date: 2/15/21

Plan of the course: define spectra and give applications.
References:

- Barnes & Roitzheim, *Foundations of Stable Homotopy Theory*
- Adams, *Stable Homotopy & Generalized Homology* (Part III)

In this lecture, we will cover four ideas leading to spectra.

1.1 Suspension

The category **Spaces** is taken to be the subcategory of 'nice' spaces in **Top**, e.g. compactly generated weakly Hausdorff spaces or simplicial sets. The category $\mathbf{Spaces}_*$ is the category of pointed spaces. We can define $[X, Y]$ = pointed homotopy classes of maps and $X \wedge Y = X \times Y/X \vee Y$. Here, S^0 is the monoidal unit.

Definition 1.1. *The **suspension** of X is*

$$\Sigma X = S^1 \wedge X = [0,1] \times X/\{0,1\} \times X \cup [0,1] \times *.$$

For example, $\Sigma S^n = S^{n+1}$.
Suspension preserves some information. For example, we have

Theorem 1.2 (Freudenthal suspension theorem)**.** *If X is a CW-complex of dimension $\leq 2n$, if Y is n-connected (i.e. $\pi_k(Y) = 0$ for $k \leq n$), then*

$$[X, Y] \xrightarrow{\cong} [\Sigma X, \Sigma Y].$$

Corollary 1.3. *If Y is n-connected, then*

$$\pi_k(Y) \xrightarrow{\cong} \pi_{k+1}(\Sigma Y)$$

for $k \geq 2$.

Then we see that if $n \geq k+2$, we have $\pi_{n+k}(S^n) \cong \pi_{n+k+1}(S^{n+1})$. For instance, for $n \geq 3$, we have that $\pi_{n+1}(S^n) \cong \mathbb{Z}/2$, which ultimately comes from the Hopf fibration.
Many other things are also stable beside homotopy groups. For instance, cohomology:

$$\tilde{H}^*(X) = \tilde{H}^{*+1}\Sigma X.$$

This isomorphism preserves the action of the Steenrod operations but not the cup product.

Example 1.4. *Consider the group of n-manifolds with some structure (complex, orientation, spin, etc.) up to bordism.*

Recall that there is a (co)bordism between two manifolds if their disjoint union is the boundary of a compact manifolds. For instance, a pair of pants gives a bordism between a circle and two circles. If $\mathrm{Bord}\, G_n$ = bordism group of n-manifolds with G-structure. Thom defined spaces $MG(n)$:

$$\mathrm{Bord}\, G_n = \mathrm{colim}_k\, \pi_{n+k} MG(k).$$

We thus see that something 'stable' is going on here.

Definition 1.5 (Spanier-Whitehead category)**.** *Objects: (X, n) where X is a pointed finite CW-complex, $n \in \mathbb{Z}$.*
Maps: between (X, n) and (Y, m): $\mathrm{colim}_k[\Sigma^{n+k} X, \Sigma^{m+k} Y]$.

In this category, $(X, n+m) \cong (\Sigma^n X, m)$. So (X, n) is a formal suspension/desuspension. In **Spaces**, there is a natural map $\pi_a(S^b) \times \pi_b(S^c) \to \pi_a(S^c)$. In **SW**,

$$\pi_* \mathbb{S} = \bigoplus_n Mor((S^0, n), (S^0, 0))$$

is a graded ring. Here $\mathbb{S}$ is the 'sphere spectrum', something we will define precisely later on.

Spectra are supposed to contain "infinite suspensions" $\Sigma^\infty X$ of any pointed space X.

Moreover, suspension is invertible in **Spectra**.

1.2 Generalized cohomology theories

A **generalized cohomology theory** is a functor $\tilde{E}^* : Spaces_*^{op} \to GrAb$ such that

1. $\tilde{E}^*$ is homotopy invariant.
2. $\tilde{E}^* X \cong \tilde{E}^{*+1} \Sigma X$, natural in X.
3. If $A \hookrightarrow X$ is an inclusion of pointed spaces, there's an exact sequence

$$\tilde{E}^*(X/A) \to \tilde{E}^*(X) \to \tilde{E}^*(A)$$

4. $\tilde{E}^*$ takes coproducts to products.

Note: we can get a long exact sequence in the following way. Recall the cofiber sequence

$$A \hookrightarrow X \to X/A \cong X \cup CA \to \Sigma A.$$

Apply the cohomology theory to get a long exact sequence.
Of course, ordinary cohomology is an example. Here is another example.

Example 1.6 (K-theory)**.** *Define $KU^0(X)$ to be the group completion of*

$$\{\textit{isomorphism classes of complex vector bundles over } X\}.$$

Reduced version: $\tilde{KU}^0(X) = \ker(KU(X) \to)$
So we can define $\tilde{KU}^{-n}(X) = \tilde{KU}^\circ(\Sigma^n X)$ for $n > 0$. How do we get to positive degree? Bott periodicity, which states

$$\tilde{KU}^0(X) \cong \tilde{KU}^0(\Sigma^2 X).$$

We use this to define $\tilde{KU}^n(X)$, and it can be checked (nontrivially) that this is a generalized cohomology theory.
We have

$$\tilde{KU}^*(S^0) = \mathbb{Z}[\beta^{\pm 1}], |\beta| = 2.$$

Cohomology theories are stable invariants. Accordingly, we get:

Every cohomology theory is represented by a spectrum.

This is the content of the **Brown representability theorem**.

Theorem 1.7 (Brown representability theorem)**.** *For any $\tilde{E}^*$, there is a spectrum E such that*

$$\tilde{E}^n(X) = [\Sigma^{-n}\Sigma^\infty X, E].$$

Note: one can do the same thing for *homology* theories.

1.3 Infinite loop spaces

Brown showed that for any $\tilde{E}^*$, there are spaces E^n such that

$$\tilde{E}^n(X) = [X, E^n].$$

For example,

$$\tilde{H}^n(X; R) = [X, K(R, n)].$$

For K-theory, we have

$$\tilde{KU}^n(X) = \begin{cases} [X, \mathbb{Z} \times BU] & n \text{ even} \\ [X, \Omega(\mathbb{Z} \times BU)] & n \text{ odd}. \end{cases}$$

Recall that

$$[X, E^n] \cong [\Sigma X, E^{n+1}] \cong [X, \Omega E^{n+1}].$$

We get by Yoneda that $E^n \cong \Omega E^n \cong \Omega^2 E^{n+2} \cong \cdots$. Thus, E^n is an **infinite loop space**.

Back to ordinary cohomology, we also have

$$K(R, n) \cong \Omega K(R, n+1) \cong \Omega^2 K(R, n+2).$$

Also, $\mathbb{Z} \times BU$ is an infinite loop space.

If X is a loop space, $X = \Omega X'$, there is a multiplication map

$$X \times X \to X$$

which is associative up to homotopy (same for inverses). This means that $[A, X]$ is a group.

If X is a *double* loop space, then $[A, X]$ is abelian.

An *infinite* loops space has an E_∞-algebra structure. This is a multiplication that is associative and commutative up to homotopy that is 'as coherent as possible'.

Theorem 1.8 (May)**.** *Let X be an E_∞-algebra and let $\pi_0(X)$ be a group. Then X is equivalent as an E_∞-algebra to an infinite loop space.*

Every infinite loop space is associated to a spectrum, and vice versa.

1.4 Formal properties of spectra

Recall the cofiber sequence in topological spaces:

$$A \to X \to X \cup CA \to \Sigma A \to \Sigma X \to \cdots .$$

These induce long exact sequences on cohomology and homology.

Dually, if we begin with a fibration, we have the fiber sequence

$$\cdots \to \Omega E \to \Omega B \to F \to E \to B.$$

These give long exact sequences on homotopy groups.

One can try to do this sort of thing in any category similar to the category of spaces; in particular, in model categories. The category of **spectra** is such a category, and moreover, fiber sequences are cofiber sequences and vice versa! Invertibility of suspension is a consequence.

Note that the *derived category* of a ring $R : \mathcal{D}(R)$, the category of chain complexes on R up to quasi-isomorphism.

The abstract framework for these subjects are *stable model categories* and *stable ∞-categories*.

Theorem 1.9 (Lurie)**.** *The category of spectra is the free presentable stable ∞-category on one object. It is also the initial presentably symmetric monoidal ∞-category.*

Example 1.10. $\mathcal{D}(R) \cong$ *Modules over* HR *inside spectra.*

Here, HR represents $\tilde{H}^*(-; R)$.

The category **Spectra** is a 'universal' example of a 'category like **Ch**(R)'.

2 Cofiber and fiber sequences

Lecturer: Paul VanKoughnett, Date: 2/15/21

Today we will review fiber and cofiber sequences in a way that will be easily amendable to what we will discuss with spectra.

References:

1. May's *A concise course in algebraic topology*
2. Hirschhorn, *The Quillen model category of topological spaces*
3. Aguilar-Gitler-Prieto, *Algebraic topology from the homotopical viewpoint*

To these I will personally add tom Dieck's *Algebraic Topology*, which I used to fill in some details.

2.1 The homotopy lifting property

Let $i : A \hookrightarrow X$ be a pointed CW-inclusion. We can form the quotient X/A; we can also form the pointed cone $C(i) = X \cup_i CA$, where $CA = A \times I/A \times \{0\} \cup * \times I$. The pointed cone is a version of the quotient that is better suited to homotopy theory.

Proposition 2.1. *There is a homotopy equivalence* $C(i) \cong X/A$*.*

Proof. There is an open neighborhood U of A in X such that U deformation retracts onto A (this uses the fact that i is a pointed CW-inclusion, and is in fact equivalent to i being a closed cofibration). This gives a map

$$C(i) \to C(i)/CA = X/A.$$

We define the homotopy inverse by the identity on $(X - U)/A$, and U interpolates between the identity and the retraction to A. □

Alternatively, we can prove this using the **homotopy extension property** (HEP).

Definition 2.2 (HEP)**.** *The map* $A \xrightarrow{i} X$ *has the HEP if*

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Maps}(I, Y) \\ \downarrow & \overset{\exists}{\nearrow} & \downarrow e^0 \\ X & \longrightarrow & Y \end{array}$$

*That is, we can extend to homotopy of A along all of X. Maps satisfying the HEP are often called **cofibrations**, but because of possible ambiguity due to multiple model structures we will just say HEP for now.*

Note: one needs to be careful to work with nice spaces (compactly generated Hausdorff?) for this to act properly.

Lemma 2.3. *CW-inclusions have HEP.*

Proof. A CW inclusion is a composition of attaching by a single cell. It suffices to show this for a single cell. In fact, we can easily reduce this to the case of $S^{n-1} \to D^n$. This can be done explicitly by constructing a retraction. □

Lemma 2.4. *Suppose we have*

$$\begin{array}{ccc} A & & \\ \downarrow{\scriptstyle i} & \searrow^{j} & \\ X & \xrightarrow{f} & Y \end{array}$$

For any map $f : A \to X$ of CW-complexes, we can factor f through the mapping cylinder $A \to X \cup_f (A \times I) \xrightarrow{g} Y$, where g is a homotopy equivalence.

Suppose $f : A \to X$ is already a CW-inclusion. Then the lemma implies it's a homotopy equivalence *under* A. Since the homotopy inverse preserves A, we can quotient by A and get a homotopy equivalence $C(f) \cong X/A$. (Another proof of Proposition 2.1.)

Now let $f : A \to X$ be a map of pointed spaces. We can find weakly equivalent CW-complexes from $\tilde{f} : \tilde{A} \to \tilde{X}$ to $f : A \to X$. Moreover, we can replace $\tilde{X}$ with the mapping cylinder X' so that $\tilde{A} \to X'$ has the HEP.

Definition 2.5. *The* ***homotopy cofiber*** *of $f : A \to X$ is $X'/\tilde{A} \cong C(\tilde{f})$. This is well-defined up to weak homotopy equivalence.*

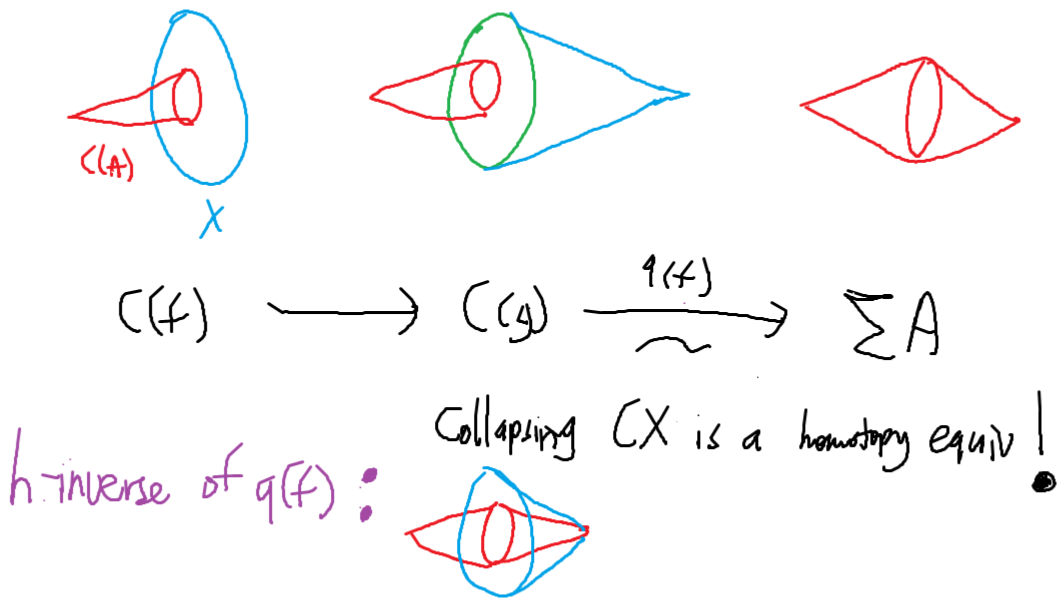
Recall that Whitehead's theorem states that a weak homotopy equivalence between CW-complexes is in fact a homotopy equivalence. Thus if we are working with CW-complexes, we recover the full homotopy type of the cofiber. Moreover, weak homotopy equivalence is in some ways a more important notino than homotopy equivalence, at least for the path we will be going. However, if we are working with all topologcal spaces as below, we will simply use the cone $C(f)$ for the homotopy cofiber.

2.2 Cofiber sequences

In both this section and the next, we will be working with pointed spaces. However, ~~due to laziness~~ for the sake of convenience we will often omit pointed notation. Now let us begin this process

$$A \xrightarrow{f} X \xrightarrow{g} C(f) \to C(g) \cong \Sigma A \xrightarrow{\Sigma f} \Sigma X \xrightarrow{\Sigma g} \cdots$$

This is known as the **Puppe sequence** (or cofiber sequence). Let us briefly explain where this comes from. If replace X with M_f, this factors f into a cofibration and a homotopy equivalence. The homotopy cofiber $C(f)$ is then homotopy equivalent to the literal quotient of the mapping cone M_f by A. On the next step, we would like to show that $C(g) \cong \Sigma A$. This is shown in the following diagram:



The homotopy inverse exhibited is the inclusion of the red ΣA into $C(g)$. As the name suggests, the homotopy cofiber is defined up to homotopy equivalence, which is why we may replace $C(g)$ with σA. But now we can iterate this process, so the next space can be taken to be ΣX, and then $\Sigma C(f)$, etc. It can also be shown that the maps $\Sigma Y \to \Sigma Z$ can be taken to be Σj, where j is the map $Y \to Z$, at any stage of this sequence.

The interest in forming this sequence comes from the following proposition, which essentially says that the Puppe sequence is **h-coexact**.

Proposition 2.6. *Let Z be any pointed space. There is a long exact sequence obtained by applying $[-, Z]^0$ onto the Puppe sequence.*

$$\cdots [\Sigma X, Z]^0 \to [\Sigma A, Z]^0 \to [C(f), Z]^0 \to [X, Z]^0 \to [A, Z]^0.$$

This is a long exact sequence of pointed sets. However, note that $[\Sigma X, Z]$ is always a group and $[\Sigma^2 X, Z]$ is always an abelian group, so these soon end up being exact sequences of (abelian) groups. We will treat these facts in more detail in the following section.

Proof. Because the sets in question are homotopy classes, it suffices to show exactness just for a single part of the sequence. Namely, we must show that

$$[C(f), Z] \to [X, Z] \to [A, Z]$$

is exact. This is very straightforward. Take $q \in [X, Z]$ such that $q|_A$ is nullhomotopic. Then q extends to $X \cup CA$ by defining q on CA to precisely be the nullhomotopy.

Conversely, suppose that q extends to $CF \to Z$. Restricting to A gives a nullhomotopy through the values of q on CA. □

By taking Z to represent some cohomology theory (e.g., $K(\mathbb{Z}, n)$ for ordinary cohomology), we get long exact sequences in cohomology. Indeed, We use the $\Sigma - \Omega$ adjunction to get

$$[\Sigma A, K(\mathbb{Z}, n)] = [A, \Omega K(\mathbb{Z}, n)] = [A, K(\mathbb{Z}, n-1)] = H^{n-1}(A, \mathbb{Z}).$$

Note that if Z is an infinite loop space, then we an extend this arbitrarily.

Note that $A \to X$ has the HEP as long as it is a relative cell complex (getting X from A by attaching cells in any way). So we can define cofibrations as retracts of relative cell complexes.

Remark. In sSets instead of **Top**: any inclusion of a simplicial subset is a cofibration that satisfies the analogue of HEP.

2.3 Fiber sequences

Let us dualize everything.

Definition 2.7. *A map $p : E \to B$ is a* ***Serre fibration*** *if lifts exist in any diagram of the following form.*

$$\begin{array}{ccc} D^n & \xrightarrow{f} & E \\ \downarrow & \overset{\exists}{\nearrow} & \downarrow{\scriptstyle p} \\ D^n \times I & \longrightarrow & B \end{array}$$

Equivalently, we may replace D^n with all CW-complexes.

Remark. If we replace D^n with all spaces, these are called Hurewicz fibrations. These are not as important for us right now.

Fiber bundles $p : E \to B$ are important examples of Serre fibrations.

Similar to before, we will want to replace maps with a composition of a homotopy equivalence and a fibration.

Definition 2.8. *(mapping path space) Given a map $f : X \to Y$, define the* ***mapping path space***

$$P_f \coloneqq \{(x, \alpha) \in X \times Y^I | \alpha(0) = f(x)\}.$$

This means we can draw a cute commutative square

$$\begin{array}{ccc} X & \xrightarrow{h-equiv} & P_f \\ \downarrow{\scriptstyle cofib.} & & \downarrow{\scriptstyle fib.} \\ M_f & \xrightarrow{h-equiv} & Y \end{array}$$

Whereas before we were replacing Y with M_f and then continuing to the right, here we are replacing X with P_f and continuing to the left. Thus, our next step is to define the homotopy fiber.

Definition 2.9 (homotopy fiber)**.** *Let the* ***homotopy fiber*** *$F(f)$ of $f : X \to Y$ be the fiber $p_1^{-1}(*)$ for an equivalent Serre fibration $p : E \to Y$. Explicitly, if we take $E = P_f$, then*

$$F(f) \coloneqq \{(x, \alpha) \in X \times Y^I | \alpha(0) = f(x), \alpha(1) = *\}.$$

Through interatively taking homotopy fibers, we obtain the **fiber sequence**:

$$\cdots \xrightarrow{\Omega h} \Omega(f) \xrightarrow{\Omega g} \Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{h} F(f) \xrightarrow{g} X \xrightarrow{f} Y.$$

Note that, as everything lives in the (pointed) homotopy category[1], we can produce the term preceding $F(f)$ by simply taking the literal fiber. Unwinding the definitions, this is ΩY. The rest of the sequence follows easily.

The fiber sequence is **h-exact**.

Proposition 2.10. *Let B be any pointed space. Then there is an exact sequence*

$$\cdots \xrightarrow{\Omega h} [B, \Omega F(f)]^0 \xrightarrow{\Omega g} [B, \Omega X]^0 \xrightarrow{\Omega f} \Omega[B, Y]^0 \xrightarrow{h} [B, F(f)]^0 \xrightarrow{g} [B, X]^0 \xrightarrow{f} [B, Y]^0.$$

From the fourth place on these are groups, and from the seventh place on these are abelian groups.

[1]Here I simply mean up to homotopy equivalence.

The proof of exactness is very straightforward, just as in the cofiber case. We just need to prove exactness for the last triple, which essentially falls out from the construction.

Letting B be a point, we get the long exact sequence of homotopy groups associated to a fibration.

Remark. If we restrict our spaces to CW-complexes, then we can replace homotopy equivalence with weak homotopy equivalence. While this may not seem fruitful right now, this perspective is (apparently) what leads to desired generalizations.

Example 2.11. *The **path space fibration** is the fibration $\Omega X \to * \to X$. Here the path space PX is replaced with $*$, which is homotopy equivalent. This can be used for computations.*

2.4 Group and cogroup structures

The functors Σ and Ω give topological spaces a group/cogroup-like structure. To be precise, we define **H-spaces**[2].

Definition 2.12. *An **H-space** is a monoid in* $\mathbf{hTop}^0$*. That is, a pointed space $(X, *)$ with a map $m : X \times X \to X$ such that $x \mapsto m(*, x)$ and $x \mapsto m(x, *)$ are homotopic to the identity.*

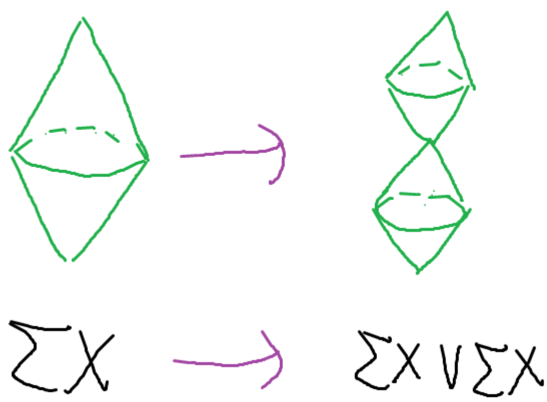
If X is any pointed space, then ΩX is an H-space. Indeed, the natural concatenation of loops forms the multiplication map. But we can say more: ΩX is a **group object** in $\mathbf{hTop}^0$. Checking this is equivalent to checking that the fundamental group really is a group.

With this group object structure, we get that $[B, \Omega X]^0$ is a group. Recall that with the fiber sequence, we stated that $[B, \Omega^2 X]$ is abelian. Indeed, if B is a point, this is simply the commutativity of $\pi_2(X)$. And this argument directly extends to showing that $[B, \Omega^2 X]$ is abelian for all B.

The dual cogroup concept for topological spaces is slightly more involved.

Definition 2.13. *A **comonoid** in* $\mathbf{hTop}^0$ *is a pointed space C along with a comultiplication map $\mu : C \to C \vee C$ such that the projections $\mathrm{pr}_1\, \mu$ and $\mathrm{pr}_2\, \mu$ are pointed homotopic to the identity.*

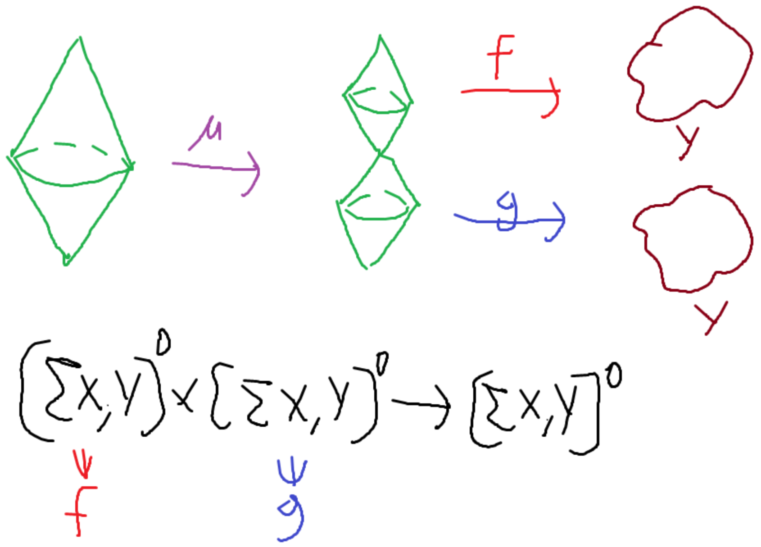
The space ΣX is a comonoid, where the map $\mu : \Sigma X \to \Sigma X \vee \Sigma X$ is given by collapsing the $t = 1/2$ portion.



[2] H for Hopf!

One can check the identity axiom for this operation. But ΣX is actually a cogroup. That is, it also satisfies coassociativity and has a coinverse. Coassociativity essentially says that $(\Sigma X \vee \Sigma X) \vee \Sigma X$ and $\Sigma X \vee (\Sigma X \vee \Sigma X)$ are pointed homotopic. The coinverse ι is given by $(x,t) \mapsto (x, 1-t)$. This satisfies the property that $\delta(\mathrm{id} \vee \iota)\mu$ and $\delta(\iota \vee \mathrm{id})\mu$ are both pointed homotopic to the constant map. Here $\delta : \Sigma X \vee \Sigma X \to \Sigma X$ is just the identity on both pieces.

The group structure on $[\Sigma X, Y]^0$ is illustrated below.



Now one can prove that $[\Sigma^2 X, Y]^0$ is abelian explicitly. Alternatively, one can use the $\Sigma - \Omega$ adjunction to translate this group into $[X, \Omega^2 Y]^0$, whose commutativity we've already shown. Well, one needs to know that the adjunction actually preserves the group structure. But this can be checked explicitly.

Remark. If (C, μ) and (M, m) are monoids in $\mathbf{hTop}^0$, then there are two composition laws on $[C, M]^0$. They must coincide and are associative and commutative. This of course can be applied to ΣX and ΩY. I believe this is a special case of the **Eckmann-Hilton argument**, and the proofs are all just bashing.

3 Spectra

Lecturer: Paul VanKoughnett, Date: 3/1/21

Today we will discuss Brown representability and begin discussing spectra.

References:

- Barnes & Roitzheim, *Foundations of Stable Homotopy Theory*
- Adams, *Stable Homotopy & Generalized Homology* (Part III)

3.1 Reduced cohomology theories

Recall that a reduced cohomology theory is a functor $\tilde{E}^* : \mathbf{CW\text{-}complexes}_*^{\mathrm{op}} \to GrAb$ satisfying

1. homotopy invariant
2. $A \overset{i}{\hookrightarrow} X \to C(i)$ gets sent to an exact sequence

$$\tilde{E}^*(C(i)) \to \tilde{E} * (X) \to \tilde{E}^*(A).$$

3. Sends $\vee$ to $\coprod$ (wedges of spaces to products of graded abelian groups)
4. There is a suspension isomorphism

$$\tilde{E}^n(X) \xrightarrow{\cong} \tilde{E}^{n+1}(\Sigma X).$$

Remark. For unpointed spaces, $E^*(X) = \tilde{E}^*(X \coprod *)$.

If $A \hookrightarrow X$ is an inclusion of CW-complexes, then $E^*(X, A) = \tilde{E}^*(X/A)$.

Theorem 3.1 (Brown representability)**.** *To any such $\tilde{E}$, there are associated spaces E_n such that*

$$\tilde{E}^n(X) \cong [X, E_n]$$

for all pointed connected CW complexes X.

Sketch. (My notes for this proof are really terrible. See Andrew Putnam's notes for a much better sketch.)

By Yoneda, there should be a class $c_n \in \tilde{E}^n(E_n)$ such that for any X and any $d \in \tilde{E}^n(X)$, there exists a unique $X \xrightarrow{p} E_n$ such that $p^*(c_n) = d$.

We define the E_n as follows. Let $E_n^0 = *$. Assume inductively that we have E_n^r such that for all $1 \leq k \leq r$, we have

$$[S^k, E_n^r] \cong \tilde{E}_n(S^k).$$

Attach $(r+1)$-spheres for each generator of $\tilde{E}_n(S^{r+1})$.

Then $\tilde{E}_n(E_n^{(r)} \vee \vee S^{r+1}) = \tilde{E}^n(E_n^r) \times \prod \tilde{E}^r(S^{r+1})$.

We get a surjection $[S^{r+1}, E_n^r \vee \vee S^{r+1}] \twoheadrightarrow \tilde{E}^n(S^r?)$.

Then we attach $r+1$-cells to kill the kernel. This defines E_n^{r+1}. Then we define

$$E_n = \bigcup_{r=0}^{\infty} E_n^r = \operatorname{hocolim} E_n^r.$$

Something about cohomology commuting with (homotopy) colimits in the right way...

So all this proves that E_n represents cohomology for spheres. Induct to show it works for finite CW-complexes. Then do another homotopy colimit argument to make it work for infinite CW-complexes.

□

What is the purpose of the suspension isomorphism? Well, we have established that

$$[X, E_n] = \tilde{E}^n(X) \cong \tilde{E}^{n+1}(\Sigma X) = [\Sigma X, E_n] = [X, \Omega E_{n+1}].$$

Thus, $E_n \cong \Omega E_{n+1}$ – a weak homotopy equivalence. This is the motivation for spectra.

Remark. These are now all abelian groups because of the Ω^2 stuff.

3.2 Defining spectra

Definition 3.2. *A **spectrum** is a sequence of pointed spaces X_n for $n \in \mathbb{N}$, with maps $\Sigma X_n \to X_{n+1}$.*

Alternatively, we can state it via maps $X_n \to \Omega X_{n+1}$.

Example 3.3. *Given any $\tilde{E}^*$, the representing spaces E_n form a spectrum with the weak homotopy equivalences $E_n \cong \Omega E_{n+1}$. This is called an **Ω-spectrum**.*

Example 3.4. *Suppose $K \in$ **Spaces**$_*$. The **suspension spectrum** of K is the spectrum $\Sigma^\infty K$ with*

$$(\Sigma^\infty K)_n = \Sigma^n K.$$

Example 3.5. *If $A \in \mathrm{Ab}$, the **Eilenberg-Maclane spectrum** has*

$$(HA)_n = K(A, n).$$

Here we have $K(A, n) \xrightarrow{\cong} \Omega K(A, n+1)$.

Example 3.6 (complex K-theory). *$KU_n = \mathbb{Z} \times BU$ for n even and U for n odd, where U is the infinite dimensional unitary group. Then $U = \Omega(\mathbb{Z} \times BU)$, and $\mathbb{Z} \times BU \xrightarrow{\cong} \Omega U$ by Bott periodicity.*

Example 3.7 (real K-theory). *$KO_{8n} = \mathbb{Z} \times BO$, $KO_{2n-1} = \Omega(\mathbb{Z} \times BO)$, etc. and the map comes from Bott periodicity which tells us that $\Omega(\mathbb{Z} \times BO) = \mathbb{Z} \times BO$.*

We will use **Sp** to denote **Spectra**. **Sp** is tensored and cotensored over **Spaces**$_*$.

Example 3.8. *Given $X \in$ **Sp** and $K \in$ **Spaces**$)*$, we define*

$$(X \wedge K)_n = X_n \wedge K.$$

Then we have a map $\sigma(X_n \vee K) = \Sigma X_n \vee K \to X_{n+1} \vee K$.

One can also define $F(K, X)_n$ to be function spaces of spectra, which are naturally spectra themselves.

Example 3.9 (Thom spectra). *Recall that $BO(n)$ classifies n-dimensional real vector bundles. Let ξ_n be the universal bundle over $BO(n)$.*

$$\mathrm{Th}(\xi_n) = \textit{disk bundle of } \xi_n/\textit{sphere bundle of } \xi_n.$$

We have the following commutative diagram.

$$\begin{array}{ccc} \xi_n \oplus \mathbb{R} & \longrightarrow & \xi_{n+1} \\ \downarrow & & \downarrow \\ BO(n) & \longrightarrow & BO(n+1) \end{array}$$

This induces a map $\Sigma\,\mathrm{Th}(\xi_n) = \mathrm{Th}(\xi_n \oplus \mathbb{R}) \to \mathrm{Th}(\xi_{n+1})$. This spectrum is called MO, the real cobordism spectrum.

3.3 Homotopy groups of spectra

Let X be a spectrum. We define

$$\pi_n(X) = \mathrm{colim}_r\, \pi_{n+r}(X_r).$$

Indeed, we have maps $\pi_{n+r}(X_r) \to \pi_{n+r}(\Omega X_{r+1}) = \pi_{n+r+1}(X_{r+1})$. Note that this is well -defined for negative n!

If E is associated to $\tilde{E}^*$, then

$$\pi_{n+r}(E_r) = [S^{n+r}, E_r] = \tilde{E}^r(S^{n+r}) = \tilde{E}^0(S^n).$$

Finally, note that $\pi_* \Sigma^\infty K = \pi_*^{\mathrm{st}} K$.

Defining maps between spectra takes quite a lot of work. Here's a 'naive' approach.

First idea: $\mathrm{Hom}(X, Y) = \{X_n \to Y_n\}$ making the maps commute.

Here is a problem with this definition. Let

$$\mathbb{S} = \Sigma^\infty S^0 = \{S^0, S^1, S^2, \ldots\}$$

and let

$$\mathbb{S}^{(1)} = \Sigma^\infty S^0 = \{*, S^1, S^2, \ldots\}.$$

Then the inclusion $\mathbb{S}^{(1)} \hookrightarrow \mathbb{S}$ induces an isomorphism on all π_*, but there is no constant map $\mathbb{S} \to \mathbb{S}^{(1)}$.

Here is another example of a problem. Consider the Hopf map $\eta : S^3 \to S^2$; this defines a nontrivial stable homotopy class, since any suspension of this map is nonzero. So this gives a map $\Sigma^\infty S^3 \to \Sigma^\infty S^2$. We can write this as

$$\Sigma^3\mathbb{S} \to \Sigma^2\mathbb{S}.$$

In **Spectra**, we should be able to desuspend things. So this should give us a map $\Sigma^1\mathbb{S} \to \mathbb{S}$, which is a map $S^1 \to S^0$ on the zero space. Such a map is constant, so if all maps commute then all the maps are constant! This is a problem.

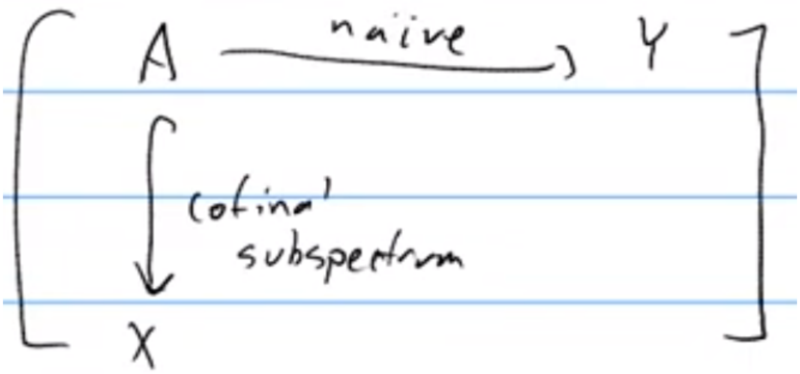
Here is Adam's solution. Restrict to CW-spectra. Maps to cells.

A **cofinal subspectrum** of X is, for each n, a subcomplex $\{A_n \hookrightarrow X_n\}$ such that

$$\begin{array}{ccc} \Sigma A_n & \longrightarrow & A_{n+1} \\ \downarrow & & \downarrow \\ \Sigma X_n & \longrightarrow & X_{n+1} \end{array}$$

commutes (all maps are inclusions). Furthermore, the cofinality condition means that for any cell $e_k \subseteq X_n$, some suspension $\Sigma^r e_k$ is in A_{n+r}.

A map $X \to Y$ is an equivalence class of diagrams



It is possible to define homotopy using $X \wedge (I \coprod *) \to Y$. One can define the stable homotopy category this way.

This comes from Adam's blue book. This is rather difficult to work with. A good alternative, which we will discuss next, is through **model categories**.

4 Model categories

Lecturer: Ivo Vekemans, Date: 3/8/21

Quillen developed model categories in 1967.

In a homotopical category, we have **weak equivalences**, which behave like isomorphisms but fail to be invertible. Under some functor, they become isomorphisms. So, what happens if we turn all the weak equivalences into isomorphisms? To make this idea work, we use model categories.

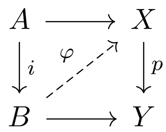
References:

- *Model Categories* by Hovey
  https://people.math.rochester.edu/faculty/doug/otherpapers/hovey-model-cats.pdf
- *More Concise Algebraic Topology* by Ponto and May
  http://www.math.uchicago.edu/~may/TEAK/KateBookFinal.pdf

4.1 Lifting and weak factorization systems

We begin with some preliminaries which will be used in the definition of model categories.

Definition 4.1. *Let $i : A \to B$ and $p : X \to Y$ in $\mathbf{M}$ be so that for all commutative squares*



*there exists $\phi : B \to X$ such that the diagram commutes. Then we say that i has the **left lifting property** wrt p and p has the **right lifting property** wrt i and we write $i \boxslash p$.*

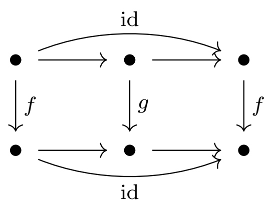
Given a collection of maps I we write

$$I^{\boxslash} = \{g \in \mathrm{Mor}(\mathbf{M}) | f \boxslash g \,\forall f \in I\}.$$

$$^{\boxslash}I = \{h \in \mathrm{Mor}(\mathbf{M}) | h \boxslash f \,\forall f \in I\}.$$

Then if $L \subseteq {}^{\boxslash}I$ we write $L \boxslash I$, and if $R \subseteq I^{\boxslash}$ we write $I \boxslash R$.

Definition 4.2. *A map f is a retract of g if there exists a commutative diagram*

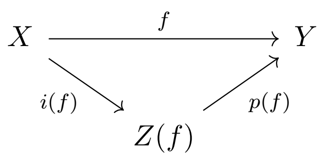


We have certain closure properties as follows. Take $I \subseteq \mathrm{Mor}(M)$. Then $I^{\boxslash}$ is closed under composition, retracts, and pullbacks and ${}^{\boxslash}I$ is closed under composition, retracts, and pushouts. These statements are proved via diagram chasing.

We can now define weak factorization systems.

Definition 4.3 (WFS)**.** *A **weak factorization system** on a category $\mathbf{M}$ is a pair of classes of map (L, R) such that*

i) For any $f : X \to Y$ in M, the following diagram exists:



such that $i(f) \in L$ and $p(f) \in R$.

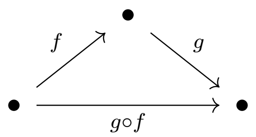
ii) $L = {}^{\boxslash}R$ and $R = L^{\boxslash}$.

Proposition 4.4. *Suppose (L, R) satisfy i) and $L \boxslash R$. If L and R are closed under retracts, then (L, R) is a WFS.*

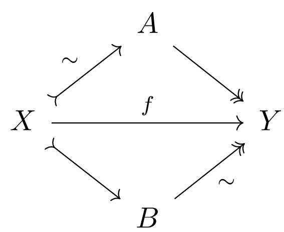
Idea of proof: factor each map and apply a retract argument.

4.2 Model categories and examples

Definition 4.5. *A **model structure** on a bicomplete category M consists of three distinguished classes of morphisms (W, C, F) with*
W = weak equivalences denoted $\xrightarrow{\sim}$
C = cofibrations denoted $\rightarrowtail$
F = fibrations denoted $\twoheadrightarrow$
satisfying the following axioms.
i) 2 out of 3: If two of



are in W, so is the third.
ii) Retracts: W, F, C are closed under retracts.
iii) Lifting: $C \boxslash (F \cap W)$ and $(C \cap W) \boxslash F$
iv) Factorization: For any $f : X \to Y$ in M we have the following factorizations.



*A **model category** is a bicomplete category M together with a model structure on M.*

Remark. Often people include functoriality into the factorization axiom, but this is not necessary.

Remark. We can replace axioms ii), iii), iv) by requiring that $(C, F \cap W)$ and $(C \cap W, F)$ are weak factorization systems.

The definition of model structure is overdetermined in the sense that if two of the three (W, C, F) are defined, then the third is determined. Indeed, we have

$$C = {}^{\boxslash}(F \cap W), \quad F = (C \cap W)^{\boxslash}.$$

Moreover, if C and F are known, then $F \cap W$ and $C \cap W$ are known. Then by the factorization and 2-out-of-3 axioms, W is determined.

We now turn to examples.

Example 4.6. *The Serre model structure on* **Top**.
W = weak homotopy equivalences
F = Serre fibtraions
C = relative cell-complexes

Example 4.7. *The Hurewicz model structure on* **Top**.
W = homotopy equivalences
F = Hurewicz fibrations
C = 'cofibrations'

Example 4.8. *Simplicial sets*

W = maps which are weak homotopy equivalences under the geometric realization functor to **Top***.*

C = maps $f : X \to Y$ such that $f_n : X_n \to Y_n$ is a monomorphism for all n.

Example 4.9. *R-Mod for R a Frobenius ring. C=injections, F=surjections.*

Example 4.10. *The projective module structre on* **Ch**$(R)_+$*, the chain complexes of R-modules concentrated in nonnegative degrees.*

W = quasi-isomorphisms, C = $F_\bullet : A_\bullet \to B_\bullet$ such that $f_n : A_n \to B_n$ is injective with projective cokernel for all n, F = morphisms that are epimorphisms in R-mod in each positive degree.

4.3 Homotopy and functors of model categories

We outline several important notions in model categories.

Definition 4.11. *X is **cofibrant** if $\emptyset \to X$ is a cofibration and **fibrant** if $X \to *$ is a fibration.*

Then Y is a cofibrant replacement for X if Y is cofibrant and there is a weak equivalence $Y \xrightarrow{\sim} X$. Similar for fibrant replacement: $X \xrightarrow{\sim} Y$. We can always find cofibrant and fibrant replacements.

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ & \searrow \quad \nearrow^{\sim} & \\ & X^{\mathrm{cof}} & \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & * \\ & {}^{\sim}\searrow \quad \nearrow & \\ & X^{\mathrm{fib}} & \end{array}$$

In Serre's **Top**, all objects are fibrant and cofibrant replacement is CW-approximation. In **sSet**, all objects are cofibrant.

Definition 4.12. *A **cylinder object** $\mathrm{Cyl}(X)$ for X is equipped with a diagram*

$$\begin{array}{ccc} X \coprod X & \xrightarrow{\nabla} & X \\ & {}_{(i_0,i_1)}\searrow \quad \nearrow_{r,\sim} & \\ & \mathrm{Cyl}(X) & \end{array}$$

via the factorization axiom we can get $(i_0, i_1) \in C$, $r \in F$, and $i_0, i_1 \in W$.

For example, in Serre's **Top**, $X \times I$ is a cylinder object for X. Cylinder objects give us left homotopies.

Definition 4.13. *A **left homotopy** between $f, g : X \to Y$ is a map $H : \mathrm{Cyl}(X) \to Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$.*

Similarly, there is a notion of path object

$$\begin{array}{ccc} Y & \xrightarrow{\Delta} & Y \times Y \\ & {}_{s,\sim}\searrow \quad \nearrow_{(e_0,e_1)} & \\ & PY & \end{array}$$

and right homotopies $H : X \to PY$ such that $e_0 \circ H = f$ and $e_1 \circ H = g$.
We say that $f, g : X \to Y$ are **homotopic**: $f \cong g$ if they are both left and right homotopic. If X is cofibrant and Y is fibrant, then $f \cong g$ is an equivalence relationon $\mathbf{M}(X, Y)$. We call an equivalence class a homotopy clas and denote them by $[f]$.

We can now define the homotopy category of M.

Definition 4.14. *Let* **M** *be a model category. The homotopy category,* $\mathrm{Ho}(\mathbf{M})$*, of* **M** *has the same objects as* **M** *and morphisms given by*

$$\mathrm{Ho}(\mathbf{M})(X,Y) = \mathbf{M}((X^{\mathrm{cof}})^{\mathrm{fib}}, (Y^{\mathrm{cof}})^{\mathrm{fib}}))/\cong .$$

Quillen functors are functors between model categories that induce derived functors between the corresponding model categories.

Definition 4.15. *Let* **M** *and* **N** *be model categories. Then a functor* $F : \mathbf{M} \to \mathbf{N}$ *is a* ***left Quillen functor*** *if it preserves* C *and* $C \cap W$*. A functor* $G : \mathbf{M} \to \mathbf{N}$ *is a* ***right Quillen functor*** *if it preserves* F *and* $F \cap W$*. An adjunction* (F, G) *is a* ***Quillen adjunction*** *if* F *is left Quillen and* G *is right Quillen.*

If $F : \mathbf{M} \to \mathbf{N}$ is left Quillen, then the **left derived functor** of F:

$$\mathbb{L}F : \mathrm{Ho}(\mathbf{M}) \to \mathrm{Ho}(\mathbf{N})$$

is given by $\mathbb{L}F(X) = F(X^{\mathrm{cof}})$.

Similarly, if $G : \mathbf{N} \to \mathbf{M}$ is right Quillen, then the **right derived functor** of G: is given by $\mathbb{R}G(X) = G(X^{\mathrm{fib}})$.

A Quillen adjunction is called a **Quillen equivalence** if $(\mathbb{L}F, \mathbb{R}G)$ is an adjoint equivalence of categories.

Example 4.16. **Top** *and* **sSet** *are Quillen equivalent via the singular chain functor and the geometric realization functor. So they have equivalent homotopy categories.*

Taking the underlying abelian groups, we get simplicial abelian groups. The ***Dold-Kan correspondance*** *is the Quillen equivalence between the category of simplicial abelian groups and the category of nonnegatively graded chain complexes. See our friend wikipedia. Better, Akhil.*

5 Cofibrantly generated model categories and the The levelwise model structure on spectra (VERY INCOMPLETE!)

Lecturer: Samuel Mercier, Date: 3/15/21

Quillen developed model categories in 1967.

References:

- *Model Categories* by Hovey
  https://people.math.rochester.edu/faculty/doug/otherpapers/hovey-model-cats.pdf
- *More Concise Algebraic Topology* by Ponto and May

5.1 Some set theory (?)

Definition 5.1 (κ-filtered ordinal)**.** *Given a cardinal* κ*, an ordinal* λ *which is - is said to be* κ*-filtered if for each* $S \subseteq \lambda$ *and* $|S| \leq k$ *then we have that* $\sup S \leq \lambda$.

Definition 5.2 (κ-small)**.** *We have a cocomplete category* **M***, a cardinal κ, and $C \subseteq \mathrm{Hom}(\mathbf{M})$ a class of maps. We say that $x \in \mathbf{M}$ is κ-small if for any κ-filtered ordinal λ and diagram $F : \lambda \to \mathbf{M}$ such that $F(\beta) \to F(\beta+1) \in C$ for all $\beta \in \lambda$, we have that*

$$\mathrm{colim}_{\alpha<\lambda} \mathrm{Hom}_{\mathbf{M}}(X, F(\alpha)) \to \mathrm{Hom}_{\mathbf{M}}(X, \mathrm{colim}_{\alpha<\lambda} F(\alpha))$$

is an isomorphism.

Example 5.3. *S^n is a class of cofibrations and in particular is ω-small in* **Spaces**$_*$*.*

Example 5.4. *An ω-small set is a finite set.*

Definition 5.5 (Cofibrantly generated model categories)**.** *A model category* **M** *is said to be* ***cofibrantly generated*** *if $\exists I, J \subseteq \mathbf{M}$ sets of maps such that*

1. *$F = rlp(J)$*
2. *$C = lp(rlo(I))$*
3. *The domains of maps in I and J are κ-small for some κ. Relative I-cell (closure under transfinite composition, pushout, and retracts) and J-cell respectively.*

Example 5.6. $I = \{S^{n-1}_+ \hookrightarrow D^n_+\}$, $J = \{D^n_+ \hookrightarrow (D^n \times I)_+\}$

This helps because we can view cofibrations as retracts of elements of I-cell. Furthermore, it gives us a model category in which factorization if functorial.

Given maps $i : X \to Y, p : X' \to Y'$, let $L(i,p)$ be the maps $g : X \to X'$, $h : Y \to Y'$ for which the diagram commutes.

... detailed construction follows ...

Define nth gluing constructions use small object argument => functorial factorization.

5.2 Sequential spectra

Morphisms of sequential spectra.

$f : X \to Y$ has these levelwise maps $f_n : X_n \to Y_n$, is a morphism of spectra if the "naive" definition works under model categories... ?

Definition 5.7 (shifted suspension spectrum)**.**

Proposition 5.8. *Cofibrantly generated model category structures*

6 Homotopy groups of spectra (MISSING)

Lecturer: Zack Garza, Date: 3/22/21

See slides and video