The Tate conjecture, BSD for function fields, Br, and Ш

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This is an overview of classical work of Tate [2] and Grothendieck [1].

1 The Tate conjecture and the finiteness of the Brauer group

Consider the cycle class map $c^r\colon Z(\overline{X}) \to H^{2r}(\overline{X}, \mathbb{Q}_l(r))$. The quotient by the kernel is defiend to be $A(\overline{X})$, and we can consider the image of $A(X)$, which lies inside the G_K-fixed part of the cohomology.

Conjecture 1.1 (Tate). *Let X be a smooth projective variety over a field k which is finitely generated over its prime field. Then the map*

$$A^r(X) \otimes \mathbb{Q}_l \to H^{2r}(\overline{X}, \mathbb{Q}_l(2r))^{G_K}$$

is an isomorphism.

Let us review some more definitions. In the case $r = 1$, there are surjections

$$\mathrm{Pic}(\overline{X}) \twoheadrightarrow \mathrm{NS}(\overline{X}) \twoheadrightarrow \mathrm{Num}(\overline{X}).$$

The kernel of the first surjection is the divisible group $\mathrm{Pic}^0(\overline{X})$, and the kernel of the second is the torsion subgroup of $\mathrm{NS}(\overline{X})$. Thus $A^1(\overline{X}) = \mathrm{Num}(\overline{X})$, and is a finitely generated abelian group with rank equal to the Picard number $\rho(\overline{X}) \coloneqq \mathrm{rank}\,\mathrm{NS}(X)$. This is finite by the theorem of the base.

Theorem 1.2. *[2, Theorem 5.2] Let X be a smooth projective geometrically connected surface over $\mathbb{F}_q$. The following statements are equivalent.*
(i) $\mathrm{Br}(X)[l^\infty]$ is finite.
(ii) The map $h\colon \mathrm{NS}(X) \otimes \mathbb{Z}_l \to H^2(\overline{X}, \mathbb{Z}_l(1))^{G_K}$ is bijective.
(iii) $\rho(X) = \mathrm{rank}_{\mathbb{Z}_l} H^2(\overline{X}, \mathbb{Z}_l(1))^{G_K}$.

Proof. It suffices to show that there is a short exact sequence

$$0 \to \mathrm{NS}(X) \otimes \mathbb{Z}_l \to H^2(\overline{X}, \mathbb{Z}_l(1))^G \to T_l(\mathrm{Br}(X)) \to 0.$$

This is because $\mathrm{Br}(X)[l^\infty]$ is finite if and only if $T_l(\mathrm{Br}(X))$ vanishes.

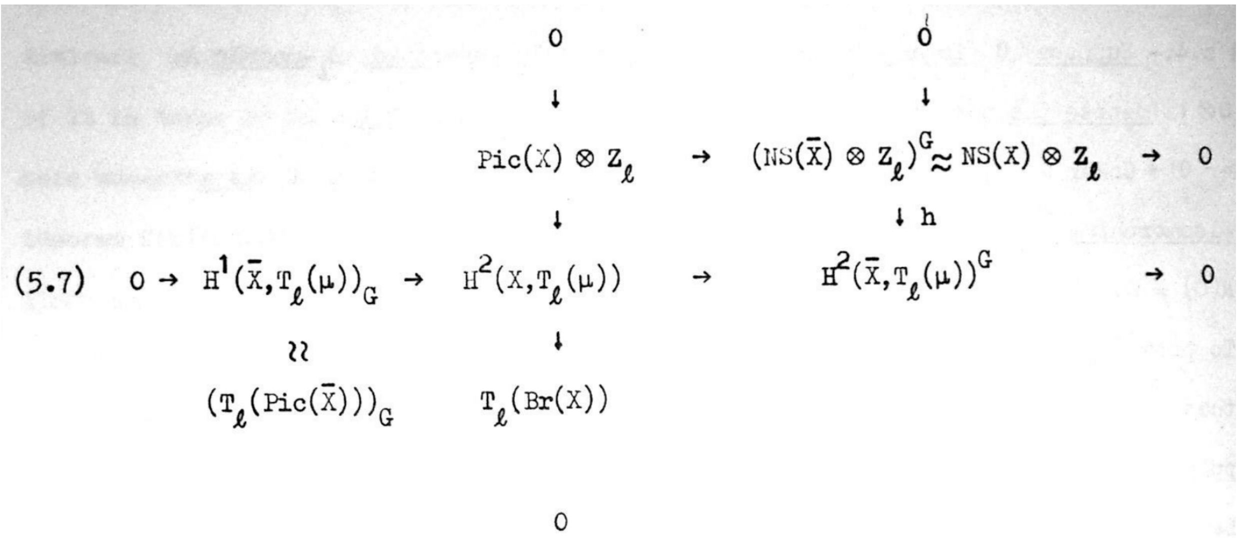
We begin with the exact sequence coming from the Kummer sequence

$$0 \to \mathrm{Pic}(X) \otimes \mathbb{Z}_l \to H^2(X, \mathbb{Z}_l(1)) \to T_l(\mathrm{Br}(X)) \to 0.$$

On the other hand, the Hochschild-Serre spectral sequence gives a short exact sequence ‘

$$0 \to H^1(\overline{X}, \mathbb{Z}_l(1))^G \to H^2(X, \mathbb{Z}_l(1)) \to H^2(\overline{X}, \mathbb{Z}_l(1))^G \to 0.$$

We put these together, and use the fact that $(\mathrm{NS}(\overline{X}) \otimes \mathbb{Z}_l)^G \cong \mathrm{NS}(X) \otimes \mathbb{Z}_l$ (this uses the fact that we are working over a finite field) to get the following commutative diagram.



The bottom left group is finite (k-points of a finite type scheme) and $T_l(\mathrm{Br}(X))$ is torsion-free. That means we can fill in the bottom right corner with $T_l(\mathrm{Br}(X))$ and the rightmost column will be exact, as desired. □

Note that these are not quite a priori equivalent to $\mathrm{Br}(X)$ being finite, because we must assume $l \neq p$ here. But it has since been shown (I believe by Milne) that the statement holds with $l = p$, so that the Tate conjecture for divisors in this case is indeed equivalent to the finiteness of the Brauer group.

2 BSD for function fields and the Tate conjecture

2.1 L-functions and zeta functions

Let $K = \mathbb{F}_q(C)$ be the function field of a smooth projective curve over $\mathbb{F}_q$. If v is a closed point of C, we write q_v for the size of the corresponding residue field. Let a_v be the trace of Frobenius at v, which is equal to $1 - |E_v(k_v)| + q_v$ if E_v is smooth, and $1, -1$, or 0 if E_v has split/non-split multiplicative reduction, or additive reduction.

Definition 2.1. *The L-function of E is defined by*

$$L(E,s) := \prod_{good\ v} \frac{1}{1 - a_v q^{(\deg v)(-s)} + q_v q^{2(\deg v)(-s)}} \prod_{bad\ v} \frac{1}{1 - a_v q^{(\deg v)(-s)}}.$$

Conjecture 2.2 (Half of BSD)**.** *Let E be an elliptic curve over a global field. Then*

$$\mathrm{ord}_{s=1} L(E,s) = \mathrm{rank}\, E(K).$$

2.2 Another equivalent statement

Now let us return to the setting of the previous section, where X is a smooth projective geometrically connected surface over $\mathbb{F}_q$. Recall that we showed that the finiteness of the Brauer group is equivalent to the Tate conjecture for divisors. It turns out that there is another equivalent statement (in fact, it is part of the same theorem in Tate's paper) that has to do with the ζ function of X.

Theorem 2.3. *[2, Theorem 5.2] In the setting above, the Tate conjecture holds for divisors, i.e. $\rho(X) = \mathrm{rank}_{\mathbb{Z}_l} H^2(\overline{X}, \mathbb{Z}_l(1))^G$ if and only if $\rho(X)$ is equal to the order of the pole of $\zeta(X,s)$ at $s = 1$.*

We recall that $\zeta(X,s) := Z(X,q^{-s}) = \exp\sum_{n\geq 1}|X(\mathbb{F}_q^n)|\frac{q^{-sn}}{n}$.

To see why this might be true, note that by the Weil conjectures, the order of the pole at $s = 1$ is the multiplicity of the eigenvalue q^{-1} of the Frobenius acting on $H^2(\overline{X},\mathbb{Q}_l)$ or equivalently the multiplicity of the eigenvalue 1 of the Frobenius acting on $H^2(\overline{X},\mathbb{Q}_l(1))$. This is at least the rank of $H^2(\overline{X},\mathbb{Z}_l(1))^G$. (Note that it is the same if the Frobenius acts semisimply!) Since we always have $\rho(X) \leq \operatorname{rank}_{\mathbb{Z}_l} H^2(\overline{X},\mathbb{Z}_l(1))^G$, we conclude. The other direction takes some more work, but is not overly difficult.

2.3 The Shioda-Tate formula

We would like to relate the Tate conjecture to BSD in the case of elliptic curves over function fields. To rpoceed we recall the Shioda-Tate formula, a key component. To even state this, one needs a nontrivial existence result, namely that of a *minimal regular proper model*. Given a smooth proper geometrically connected curve $C/\mathbb{F}_q$, and a smooth proper geometrically connected curve $X \to \mathbb{F}_q$, this is given by an appropriate surface $\mathcal{X} \to C$ whose generic fiber is X.

We will be more precise in the case of an elliptic curve $E/K = \mathbb{F}_q(C)$. Up to isomorphism there is a unique surface $\mathcal{E}/K$ with a morphism $\pi\colon \mathcal{E} \to C$ that is smooth, absolutely irreducible, and projective, with π surjective and relatively minimal, with the generic fiber equal to E/K. Relative minimality essentially means that in the fibers of π, there are no rational curves of self-intersection -1 (which can be blown-down by Castelnuovo's criterion).

Theorem 2.4 (Shioda-Tate). *In the above setting, let f_v denote the number of components of the fiber of $\mathcal{E}$ at the closed point v of C. Then*

$$\operatorname{rank} E(K) = \operatorname{rank}\operatorname{NS}(\mathcal{E}) - 2 - \sum_v (f_v - 1).$$

2.4 Equivalence for elliptic curves over function fields

Let us explain what the Shioda-Tate formula buys us. First, consider the case that $\pi\colon \mathcal{E} \to C$ is smooth, so that $f_v = 1$ for all v. By the definition of the L-function, in this case we have

$$L(E,s) = \frac{\zeta(C,s)\zeta(C,s-1)}{\zeta(\mathcal{E},s)}.$$

Since $\zeta(C,s)$ has simple poles at 0 and 1, we have that $\operatorname{ord}_{s=1} L(E,s) = \operatorname{ord}_{s=1}(\zeta(\mathcal{E},s))^{-1} - 2$.

More generally, after we account for the singular fibers at the bad primes, we have

$$\operatorname{ord}_{s=1} L(E,s) = \operatorname{ord}_{s=1}(\zeta(\mathcal{E},s))^{-1} - 2 - \sum_v (f_v - 1).$$

Comparing this to the Shioda-Tate formula, and recalling that the Tate conjecture for $\mathcal{E}$ is equivalent to $\operatorname{rank}\operatorname{NS}(\mathcal{E}) = \operatorname{ord}_{s=1}(\zeta(\mathcal{E},s))^{-1}$, we conclude:

Theorem 2.5. *The Tate conjecture holds for $\mathcal{E}$ if and only if BSD holds for E.*

Note that we also have from this argument that $\operatorname{rank} E(K) \leq \operatorname{ord}_{s=1}(\zeta(\mathcal{E},s))^{-1}$.

3 The Brauer group and the Tate-Shafarevich group

Recall the Tate-Shafarevich group of an abelian variety.

Definition 3.1. *Let A be an abelian variety over a global field K. The Tate-Shafarevich group of A is defined as*

$$Ш(A) := \ker H^1(K, A) \to \prod_v H^1(K_v, A).$$

In [1, Section 4], Grothendieck proved the following theorem.

Theorem 3.2. *In the previous setting,*

$$\mathrm{Br}(\mathcal{E}) \cong Ш(E/K).$$

The result Grothendieck proved is more general, relating the Tate-Shafarevich group $Ш(X/K)$ of a curve over a function field to the Brauer group $\mathrm{Br}(\mathcal{X})$of (one of) its regular proper models. We will give a brief sketch of the proof of the part of this result showing how if one is finite, the other is.

The starting point is the spectral sequence associated to $f\colon \mathcal{X} \to C$. This is given by

$$E_2^{pq} = H^p(C, R^q f_* \mathbb{G}_m) \Rightarrow H^{p+q}(\mathcal{X}, \mathbb{G}_m).$$

Now we use a vanishing theorem due to Artin, which says that $R^i f_* \mathbb{G}_m = 0$ for $i \geq 2$. This simplifies the spectral sequence greatly. Moreover, we will assume that X/K has a rational point, so f has a section. Then $H^2(C, \mathbb{G}_m) = 0$ and $H^3(C, \mathbb{G}_m)$ injects into $H^3(\mathcal{X}, \mathbb{G}_m)$, which ends up giving us $H^2(X, \mathbb{G}_m) \cong H^1(C, R^1 f_* \mathbb{G}_m)$.

Note that $R^1 f_* \mathbb{G}_m$ is the relative Picard functor $\mathrm{Pic}_{\mathcal{X}/C}$. To analyze this, we use a result of Raynaud, that states that $\mathrm{Pic}^0_{\mathcal{X}/C} \cong \mathcal{J}^0$, where $\mathcal{J}$ is the Neron model of $J_{X/K}$. By comparing the cohomology of $\mathrm{Pic}, \mathrm{Pic}^0, \mathcal{J}, \mathcal{J}^0$, we can conclude that $\mathrm{Br}(\mathcal{X})$ is finite if and only if $H^1(C, \mathcal{J})$ is. Finally, using results of Mazur one sees that $H^1(C, \mathcal{J})$ is finite if and only if $Ш(J)$ is.

References

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