# Hilbert's Nullstellensatz and Syzygy Theorems

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# Outline

1. History and Preliminaries
2. Noether Normalization
3. Statement and Proof of the Nullstellensatz
4. The Syzygy Theorem

# Some History

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# Mathematical Preliminaries: Definitions

- Affine Space: $K^n$ with no structure for some field K (i.e. just points in K with n coordinates)
- Zero Set: For an ideal of polynomials $J \in K[x_1, ..., x - n]$, $V(J) := \{(a_1, ..., a_n) \in K^n | f(a) = 0 \forall f \in J\}$
- Ideal of subset of Affine Space: For $X \subseteq K^n, I(X) := \{f \in K[x_1, ..., x_n] | f(x) = 0 \forall x \in X\}$
- Noetherian Ring: A ring such that every ascending chain of ideals eventually stabilizes (there is a maximal ideal in the chain)
- Algebra: A vectorspace with (bilinear) multiplication defined between vectors

# Mathematical Preliminaries: Results

**Definition:** Maximal Ideal: In a ring R an ideal m is maximal if $\nexists m'$ such that m' is a proper ideal containing m (i.e. $\nexists m'$ st $m \subsetneq m' \subsetneq R$)

**Theorem:** For a ring R and an ideal m, m is maximal if and only if R/m is a field

# Introductory Concepts for Noether Normalization

Finitely generated Algebra vs Finite Algebra:
Suppose A is an algebra over B. Then A is
finitely generated over B if $\exists a_1, ..., a_n \in A$ st $A = B[a_1, ..., a_n]$
a finite algebra over B if $\exists a_1, ..., a_n \in A$ st $A = a_1 B + ... + a_n B$

# Statement of Noether Normalization

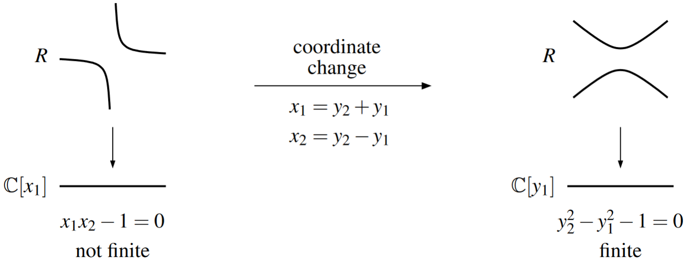
**Theorem:** Suppose K is a field. Let $A = K[a_1, ..., a_n]$ be a finitely generated K-algebra. Then $\exists y_1, ..., y_m \in A, m \leq n$ st:

1. $y_1, ..., y_m$ are algebraically independent over K
   $(\nexists 0 \not\equiv f \in K[x_1, ..., x_n]$ st $f(y_1, ..., y_m) = 0)$
2. A is a finite $K[y_1, ..., y_m]$ algebra
   i.e. If $B = K[y_1, ..., y_m]$ then $A = c_1 B + ... + c_l B$ for some $c_i \in A$

# Geometric Interpretation of Noether Normalization

Noether normalization considers algebraic sets as (finite) covers of Affine Space

**Example:**

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# Geometric Interpretation of Noether Normalization

More formally, let X be a variety in $A^n_K$, and assume X is irreducible.
Consider $A = K[a_1, ..., a_n] = K[x_1, ..., x_n]/I(X) \ \ (a_i = x_i \mod I)$.
Then $\exists$ algebraically independent $y_1, ..., y_m$ st A is a finite $K[y_1, ..., y_m]$ algebra.
Then we can define a projection (i.e. surjective map) $\phi : X \to K^m$ st $\forall z \in K^m, \phi^{-1}(\{z\})$ is finite

# The Nullstellensatz Formally Stated

**Theorem:** Let K be algebraically closed, $A = K[x_1, ..., x_n]$. Then:

1. Every maximal ideal m in A is of the form $m = (x_1 - a_1, ..., x_n - a_n) = I(P), P \in K^n$
2. If $J \subsetneq A$ is a proper ideal, then V(J)=0
   Equivalently, any set of polynomials $f_1, ..., f_m \in K[x_1, ..., x_n]$ has a common zero unless $\exists g_1, ..., g_m \in K[x_1, ..., x_n]$ st $g_1 f_1 + ... + g_m f_m = 1$
3. For every ideal $J \subseteq A, I(V(J)) = \sqrt{J}$
   That is,
   $f(x_1, ..., x_n) = 0 \forall (x_1, ..., x_n) \in V(J) \subseteq K^n \iff f^r \in J$ for some $r \in \mathbb{N}$

# Nullstellensatz Proof Ingredients

**Theorem:** Let K be an infinite field and $A = K[a_1, ..., a_n]$ a finitely generated K-Algebra. If A is also a field, then A is algebraic over K (any element in A is a root of polynomials over K).

Recall: R/m is a field if and only if m is a maximal ideal.

Recall: Every proper ideal in a Noetherian ring is contained in some maximal ideal.

# Immediate Corollaries of the Nullstellensatz

**Corollary 1.17.** *For $A = k[x_1, \ldots, x_n]$, the maps $V$ and $I$*

$$\{ideals\ of\ A\} \xleftrightarrow{V,I} \{subsets\ of\ \mathbb{A}_k^n\}$$

*induce the following bijections:*

$$\begin{array}{ccc}
\{radical\ ideals\ of\ A\} & \xleftrightarrow{1:1} & \{subvarieties\ of\ \mathbb{A}_k^n\} \\
\cup & & \cup \\
\{prime\ ideals\ of\ A\} & \xleftrightarrow{1:1} & \{irreducible\ subvarieties\ of\ \mathbb{A}_k^n\} \\
\cup & & \cup \\
\{maximal\ ideals\ of\ A\} & \xleftrightarrow{1:1} & \{points\ of\ \mathbb{A}_k^n\}.
\end{array}$$

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# Pre-Syzygy: Modules

The Syzygy theorem primarily concerns modules and their generators.
Think of modules as generalizations of vector spaces, but instead of being over fields they're over rings.
Vector spaces have bases, modules have generators which may not be independent.
Free module: A module with a basis (i.e. linearly independent generators).
We will mostly work with the free module $R^n = R \times ... \times R$ for some ring R, which is analogous to the vectorspace $K^n$ for some field K.

# Syzygys

**Definition:** A module M is said to be finitely generated over a ring R if $\exists z_1, ..., z_n \in M$ st $\forall x \in M, x = m_1 r_1 + ... + m_n r_n,$ for $r_i \in R$
The $z_i$ are the generators of M
**Definition:** A syzygy is a set of $(a_1, ..., a_n \in R^n$ st $a_1 z_1 + ... + a_n z_n = 0$ for the $z_i$ generators of a finitely generated module M over R.
**Proposition:** A syzygy is a submodule of $R^n$
We can encode syzygys as kernels of maps from free modules. This is best illustrated by example

# Syzygy Theorem

**Theorem:** Let $R = K[x_1, ..., x_n]$. Then every finitely generated R module has a free resolution of length $\leq n$.

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