zen

Chapter 2 : Intersection Multiplicities

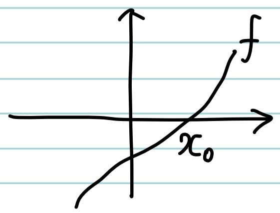
Introduction

<multiplicity of zeros>

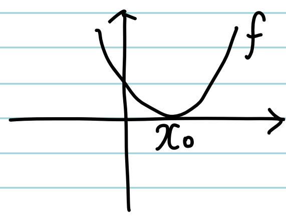
When we talk about a polynomial, like $f(x)$, having a zero at a point x_0, we can say how many times it touches the x-axis at that point. This is called the "multiplicity" of the zero

ex) • If $f(x)$ crosses the x-axis at x_0 (like a straight line), we say it has a multiplicity of 1.

• If $f(x)$ just touches the x-axis and turns around (like a parabola), we say it has a multiplicity of 2 or higher.



$f = (x-x_0)g$
multiplicity 1



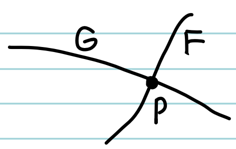
$f = (x-x_0)^2 g$
multiplicity 2

<intersection of curves>

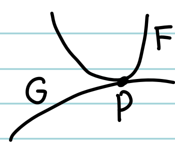
Now, instead of just one curve, we can look at two curves F and G on a graph. The intersection point P is where these two curves meet.

• If F and G cross each other at P with different angles, w say their intersection multiplicity is 1. → means they are intersecting nicely, like two straight lines crossing.

• If they are touching at the same angle, or if they both have similar slopes at P, then we say they have a higher intersection multiplicites. (2 or more)



multiplicity 1



multiplicity 2

→ if the graph flattens out at the x-axis (like a "U" shape), it's touching the axis, meaning the multiplicity is at least 2.

<singular points>

Sometimes, curves can have points where they behave weirdly, like being flat or crossing over themselves. These points are called singular points. At these points, it's not straightforward to determine the intersection multiplicity just by looking.

<constructing I·M>

To analyze how curves intersect, we need to develop a way to measure this intersection for any two curves, not just simple cases. This is what the concept of intersection multiplicity aims to achieve → For this we need the following algebraic object that allows us to capture the local geometry of the plane around a point.

<Def 2.1>

Definition 2.1 (Local rings of $\mathbb{A}^2$). Let $P \in \mathbb{A}^2$ be a point.

(a) The local ring of $\mathbb{A}^2$ at P is defined as

$$\mathcal{O}_P : \mathcal{O}_{\mathbb{A}^2, P} := \left\{ \frac{f}{g} : f, g \in K[x,y] \text{ with } g(P) \neq 0 \right\} \subset K(x,y).$$

(b) It admits a well-defined ring homomorphism

$$\mathcal{O}_P \to K, \ \frac{f}{g} \mapsto \frac{f(P)}{g(P)}$$ which we will call the evaluation map.

Its kernel will be denoted by

$$I_P := I_{\mathbb{A}^2, P} := \left\{ \frac{f}{g} : f, g \in K[x,y] \text{ with } f(P) = 0 \text{ and } g(P) \neq 0 \right\} \subset \mathcal{O}_P.$$

<In this definition,>

- $\mathcal{O}_P$ represents the local ring at point P.
- f and g are polynomial from the polynomial ring $K[x,y]$ where K is the field over which the polynomials defined
- The condition $g(P) \neq 0$ ensures that the denominator does not vanish at the point P, allowing us to evaluate these fractions near P.

The local ring O_P contains functions that behave nicely near P. It allows us to focus specifically on those functions that remain defined and provide meaningful values around the point we are interested in.

By using local rings, we can analyze the intersection multiplicity $\mu_P(F,G)$ more rigorously. The dimension of the quotient $\frac{O_P}{\langle F,G\rangle}$ gives us a precise measure of how F and G behave at P.

<Remark 2.2>

recall that the local ring at a point P in the affine plane A^2 is defined as $O_P = \{\frac{f}{g} : f,g \in K[x,y] \text{ with } g(P) \neq 0\}$

This set contains fractions of polynomials where the denominator does not vanish at point P.

<geometric interpretation>

- The remark states that O_P describes "nice" functions around point P. $\longrightarrow$ means that the functions in O_P are well defined and behave smoothly in the neighborhood of P.
- In simpler terms, these functions are reliable for calculation and visualizations near P.

- The local ring helps us focus on a small area around P. This is important because the global behavior of a curve may not reflect how it behaves right at the intersection or nearby.

<algebraic intersection>

- The remark mentions that O_P is a subring of $K(x,y)$, where $K(x,y)$ is the field of rational functions. This means that every element of O_P is a rational function, but only those that are well-defined at P.
- Being a subring implies that it contains 0 and 1 and is closed under addition and multiplication, which are essential properties for a ring.

- The local ring O_P is also an integral domain. $\rightarrow$ means it has no zero divisors. In simpler terms, if the product of two elements is zero, at least one of those elements must be zero.
- This is crucial property for ensuring that the functions we are working with have meaningful interactions.

<units and irreducible elements>

- The remark states that the units in O_P are precisely the fractions $\frac{f}{g}$ for which both f and g are non-zero at P.

Units are elements that have a multiplicative inverse in the ring, which helps in understanding which functions are invertible in this context.

- The remark also notes that O_P contains irreducible elements, which are polynomials that cannot be forced into simpler polynomials in the rings. Those irreducible polynomials vanish at P and let us know the local behavior of the curves at that point.

Finally, it emphasizes that O_P is a local ring in the algebraic sense, meaning it contains exactly one maximal ideal, denoted as I_P.

This means all other ideals in O_P either are contained in I_P or equal O_P itself. The maximal ideal consists of functions that vanish at P.

<Def 2.3>

For a point $P \in \mathbb{A}^2$ and two curves (or polynomials) F and G we define the intersection multiplicity of F and G at P to be

$$\mu_P(F,G) := \dim O_P/\langle F,G\rangle \in \mathbb{N} \cup \{\infty\},$$

where dim denotes the dimension as a vector space over K.

<In this definition,>

$\mu_P(F,G)$: represents the intersection multiplicity of the curves F and G at the point P. It is the key quantity we aim to compute.

$\langle F,G\rangle$: denotes the ideal generated by the curves F and G within the local ring O_P. This ideal consists of all functions that can be expressed as combinations of F and G with coefficients in O_P.

$\dim O_P/\langle F,G\rangle$: refers to the dimension of the quotient gives us insight into how the curves intersect at point P.

- If $\mu_P(F,G) = 1$, it indicates F and G intersect transversely at P (they cross nicely)
- If $\mu_P(F,G) > 1$, it suggests that the curves touch or overlap in a more complicated way at that point.

⟨Remark 2.4⟩

(a) It is clear from the definitions that an invertible affine transformation from (x,y) to

$$(x', y') = (ax+by+c,\ dx+ey+f) \text{ for } a,b,c,d,e,f \in K \text{ with } ae-bd \neq 0$$

gives us an isomorphism between the local rings $\mathcal{O}_P$ and $\mathcal{O}_{P'}$, where P' is the image point of P; and between $\mathcal{O}_P/\langle F,G\rangle$ and $\mathcal{O}_{P'}/\langle F',G'\rangle$ where F' and G' are F and G expressed in the new coordinates x' and y'. We will often use this invariance to simplify our calculations by picking suitable coordinates, e.g. such that $P=0$ is the origin.

- This property states that if you change the coordinates in a way that preserves the structure of the space (by using an invertible linear transformation), the local rings at the original point P and the transformed point P' will be isomorphic.

- In simpler terms, if you change the way you look at the curves, (by shifting or rotating the axes), the local behavior near the intersection remains the same.

(b) The intersection multiplicity is symmetric : we have $\mu_P(F,G) = \mu_P(G,F)$ for all F and G.

- This property highlights that the order of the curves F and G does not matter when calculating intersection multiplicity. Whether you look at how F intersects G or how G intersects F, the multiplicity will be the same. This is a reflection of the symmetric nature of intersections in geometry.

(c) For all F, G, H we have $\langle F, G+FH\rangle = \langle F, G\rangle$, and thus $\mu_P(F, G+FH) = \mu_P(F,G)$

- This property indicates that if you add a multiple of one curve to another (in this case, $G+FH$), the intersection multiplicity with respect to a third curve F remains unchanged. Essentially, adding more complexity to one curve does not affect how it intersects with another curve at point P.

<Lemma 2.5>

Let $P \in \mathbb{A}^2$, and let F and G be two curves (or polynomials).

We have: (a) $\mu_P(F,G) \geq 1$ if and only if $P \in F \cap G$

(b) $\mu_P(F,G) = 1$ if and only if $\langle F,G \rangle = I_P$ in $\mathcal{O}_P$.

(a) → means if the intersection multiplicity is non-zero, it confirms that the point P must be on both curves. If P is not on either curve, they cannot intersect there, and thus the multiplicity is zero.

(b) The second part states that $\mu_P(F,G) = 1$ iff the ideal generated by F and G, denoted as $\langle F,G \rangle$, is equal to the ideal I_P in the local ring $\mathcal{O}_P$.

→ This indicates that for the intersection multiplicity to be exactly one, the only functions that vanish at P must be those generated by the curves F and G. i.e., at P, F, and G touch each other just once without overlapping further. This represents a simple intersection where the two curves cross at a single point

Proof: Assume first that $F(P) \neq 0$. Then F is a unit in $\mathcal{O}_P$, and thus $\langle F,G \rangle = \mathcal{O}_P$. i.e., $\mu_P(F,G) = 0$.
moreover, we then have $P \notin F$ and $F \notin I_P$, proving both (a) and (b) in this sense. Of course, the case $G(P) \neq 0$ is analogous.

So we may now assume that $F(P) = G(P) = 0$. i.e., $P \in F \cap G$. Then the evaluation map at P induces a well-defined and surjective map $\mathcal{O}_P / \langle F,G \rangle \to K$.

It follows that $\mu_P(F,G) \geq 1$, proving (a) in this case. Moreover, we have $\mu_P(F,G) = 1$ if and only if this map is an isomorphism. i.e., if and only if $\langle F,G \rangle$ is exactly the kernel I_P of the evaluation map.