November 15, 2024

1. History

- Riemann zeta function
- harmonic series $\zeta(1)$; divergent, 14^{th} century
- 18^{th} century, Euler, $\zeta(2)$
- $\zeta(2n)$ $n \in \mathbb{N}$
- 19^{th} century, Riemann, ζ as complex
- 1859 paper
- Riemann hypothesis

2. Definition and Basic Properties

- $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} = \sum_{n=1}^{\infty} n^{-x}$
- $\zeta(1)$ and $\zeta(2)$
- $x^a < x^b \Rightarrow \zeta(a) > \zeta(b)$
- domain of ζ as $\{s \in \mathbb{C} \mid Re(s) > 1\}$
- Proof in lecture

3. Euler Product

- $\zeta(s) = \prod_{k=1}^{\infty} \frac{1}{1-p_k^{-s}}$
- Proof in lecture
- infinitude of the prime numbers
- Proof in lecture

difficulty of evaluation of infinitude product over all prime numbers

- ζ study using summation definition

4. Evaluating Zeta at Particular Points

- consideration of computer
- Weierstrass Factorization Theorem
- $f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$
- example with sin s.t. $\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$

- evaluation of $\zeta(2) = \frac{\pi^2}{6}$ using $\frac{\sin(x)}{x}$

- Proof in lecture

- $\zeta(2n) = \frac{|B_n| 2^{2n-1} \pi^{2n}}{(2n)!}$ for $n \in \mathbb{N}$

- $B_x = \sum_{k=0}^{\infty} \frac{1}{k+1} \sum_{r=0}^{k} (-1)^r \binom{k}{r} r^x$

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x	$\zeta(x)$
2	$\frac{\pi^2}{6}$
3	1.2020569...
4	$\frac{\pi^4}{90}$
5	1.0369277...
6	$\frac{\pi^6}{945}$

5. Analytic Continuation

- alternating zeta function

- $Z(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \dots$

- proposition of expanding the domain of ζ to $\text{Re}(s) > 0$, $s \neq 1$

- Proof in lecture

- $\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$

6. Riemann Hypothesis

- claim of all non-trivial zeros of ζ have real part of $\frac{1}{2}$

- powerful consequences

- distribution of prime numbers

7. Brief Overview of Different Types of Zeta Functions

- Dirichlet L-functions

- Artin L-functions

- Dedekind zeta functions

- Zeta functions of schemes (especially over finite fields)

- "twisted zeta function"

- $\zeta_K(s) = \sum_{I \subseteq O_K} \frac{1}{(N_{K/\mathbb{Q}}(I))^s}$

- $\zeta_K(s) = \prod_{P \subseteq O_K} \frac{1}{1 - N_{K/\mathbb{Q}}(P)^{-s}}$, $\text{Re}(s) > 1$

- $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2)$

- $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\,\Gamma(s)$
- $\Lambda_K(s) = |\Delta_K|^{s/2}\,\Gamma_{\mathbb{R}}(s)^{r_1}\,\Gamma_{\mathbb{C}}(s)^{r_2}\,\zeta_K(s)$
- $\Xi_K(s) = \frac{1}{2}(s^2 + \frac{1}{4})\,\Lambda_K(\frac{1}{2} + is)$
- functional equation
- $\Lambda_K(s) = \Lambda_K(1-s)$
- $\Xi_K(-s) = \Xi_K(s)$
- $\lim_{s \to 0} s^{-r}\zeta_K(s) = -\frac{h(K)R(K)}{w(K)}$
- $r = r_1 + r_2 - 1$
- arithmetically equivalent fields
- arithmetic zeta function
- $\zeta_X(s) = \prod_x \frac{1}{1 - N(x)^{-s}}$
- $\zeta_X(s) = \frac{1}{1-q^{-s}}$
- $\zeta_X(s) = Z(X, q^{-s})$
- $Z(X,t) * Z(Y,t) = Z(X \times Y, t)$
- $*$ is multiplication in the ring $W(\mathbb{Z})$ of Witt vectors of the integers
- zeta functions of disjoint unions
- meromorphic continuation
- generalized Riemann hypothesis