Peter-Weyl Theorem

(Chapters 4.5-4.7 of An Introduction to Lie Groups and Lie Algebras by Alexander Kirillov Jr.)

- We first want to show that a large class of representations (Unitary Representations) is $\boxed{\text{Completely reducible.}}$ (also called "semisimple")

Def A representation is called completely reducible (or semisimple) if it is isomorphic to a direct sum of irreducible representations: $V \cong \bigoplus V_i$, V_i irreducible. (only has subrepresentations 0 and V_i)

(Note: not all representations are completely reducible.

Example:

Find Example (Def 4.18 on pg. 65 of pdf)

Def A complex representation V of a real Lie group G is called Unitary if there is a G-invariant inner product: $(\rho(g)v, \rho(g)w) = (v,w)$ or equivalently, $\rho(g) \in U(V)$ $\forall g \in G$.

Definition on a Lie Group

Let's try to unpack this definition as there are a lot of new terms that, although were previously covered, can still be hard to keep track of.

Def/ A complex representation V of a Real Lie Group G is a vector space V together with a morphism $p: G \to GL(V)$, where $V \cong \mathbb{C}^n$ and G is a real Lie Group. i.e. that $p(gh) = p(g)p(h)$ and $p(1) = 1$.

[Analagous to what one may think of a standard group homomorphism)

G is a real lie group means that G has two structures, G is a group and is a manifold. (as opposed to a complex-analytic manifold) [the lie group is over real numbers].

Def/ A representation V of a real Lie Algebra $\mathfrak{g}$ is called unitary if there is an inner product which is $\mathfrak{g}$-invariant: $(p(x)v, w) + (v, p(x)w) = 0$ or equivalently, $p(x) \in U(V)$ $\forall x \in \mathfrak{g}$.

Definition on a Lie Algebra

Example: Let $V = F(S)$, which is the space of complex-valued functions on a finite set S.
Let G be a finite group acting by permutations on S, then it also acts on V by (2.1).
Then, $\langle f_1, f_2 \rangle = \sum_S f_1(s) \overline{f_2(s)}$. is an invariant inner product so such a representation is unitary.

So what?

This enables us to prove that...

Thm Each unitary representation is completely reducible.

Pf You have a set of various cases:

1) V is irreducible $\Rightarrow$ V is completely reducible by definition.

2) V has a subrepresentation W
$\Rightarrow V = W \oplus W^{\perp}$ and $W^{\perp}$ is a subrepresentation as well.

By induction, we can show that at

(still needs work)

Thm Any representation of a finite group is unitary.

Pf Let $\tilde{B}(v,w) := \frac{1}{|G|} \sum_{g \in G} B(gv, gw)$

$$\tilde{B}(hv, hw) = \frac{1}{|G|} \sum_{g \in G} B(ghv, ghw) = \frac{1}{|G|} \sum_{g' \in G} B(g'v, g'w)$$

With these two theorems, $\Rightarrow$ every representation of a finite group is completely reducible.

Goal: Generalize Complete Reducibility to larger and larger spaces.

- In our proof that representations of finite groups are completely irreducible, the strategy was to take an average for our function.
  - This works well in finite groups.

But what if G is a Lie Group?
  - Replace with an integral over G.

__Def__ A right Haar measure on a real lie group G is a Borel measure dg which is invariant under the right action of G on itself

Right invariance $\Leftrightarrow \int f(gh)\,dg = \int f(g)\,dg$ $\forall h \in G$ and integrable function f.

Left invariance $\Leftrightarrow \int f(hg)\,dg = \int f(g)\,dg$

Construction of the Haar Measure:

- Too Complicated.
  - Requires knowledge of Measure Theory and of differential forms.

(Neither of which I have a background in)

Thm Let G be a real Lie Group.

1) G is orientable; moreover, the orientation can be chosen so that the right action of G on itself preserves the orientation.

2) If G is compact, then for a fixed choice of right-invariant orientation on G, there exists a unique right-invariant top degree differential form ω such that $\int_G \omega = 1$.

3) The differential form ω defined in the previous part is also left-invariant and invariant up to a sign under $i: g \to g^{-1}$: $i^*\omega = (-1)^{\dim G}\omega$.

Thm Let G be a compact real lie group. Then it has a canonical Borel measure dg which is both left and right invariant and invariant under $g \mapsto g^{-1}$ and which satisfies $\int_G dg = 1$. This measure is called the Haar measure on G and is usually denoted by dg.

Remark: Bi-invariant Haar Measure exists in not just every Lie Group but any compact topological group (with some restrictions), [though this is also beyond the material of this talk]

Haar Measure Examples:

- Let $G = S^1 = \mathbb{R}/\mathbb{Z}$.

  Haar Measure $dx = \mathbb{R}/\mathbb{Z}$.

In general, writing an explicit Haar Measure for a group is difficult.

- Usually complicated to write out a formula for.
- Only really doable when integrating conjugation-invariant functions (Class Functions)

Example: $G = U(n)$ f smooth function on G. such that $f(ghg^{-1}) = f(h)$.

$$\int_{U(n)} f(g)\,dg = \frac{1}{n!}\int_T f\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \prod_{i<j} |t_i - t_j|^2 dt.$$

$T := \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix},\ t_k = e^{i\varphi_k} \right\}$, is the subgroup of diagonal matrices and $dt = \frac{1}{(2\pi)^n} d\varphi_1 \dots d\varphi_n$ is the Haar Measure on T.

[Clearly these examples can get very messy very fast]

Important Theorem

Thm] Any finite-dimensional representation of a compact Lie Group is unitary and thus completely reducible.

Pf) Let $B(v,w)$ be a positive definite inner product in V.

$$\tilde{B}(v,w) = \int_G B(gv, gw)\, dg,$$

dg is the Haar Measure on G.

It is clear that $\tilde{B}(v,v) = \int_G B(gv, gv) dg > 0$ since $B(gv, gv) > 0$. Because Haar measures are right-invariant, $\tilde{B}(hv, hw) = \tilde{B}(v, w)$.
By definition, this inner product is an invariant inner product, making this representation unitary.
Hence, the representation is completely reducible. $\square$

We have now gotten to a point where we have truly broaden the scope of completely reducible representations.
Obviously, we have just shown that any finite-dimension representation of a compact Lie Group is completely reducible.

In practice, how do we do an explicit decomposition into $\bigoplus n_i V_i$, $n_i \in \mathbb{Z}_+$, V_i are pairwise non-isomorphic irreducible representations

From now on, assume G is a compact real Lie Group with Haar Measure dg.

Let v_i be a basis in representation V.
$\rho(g): V \to V$ in the basis v_i yields a matrix-valued function on G.
Each entry, which we will denote as matrix coefficients of the representation V, can be denoted by scalar-valued functions on G $\rho_{ij}(g)$ in each entry.

Important Facts: (Won't prove for sake of time)

(1) Let V, W be non-isomorphic irreducible representations of G. Choose bases $v_i \in V$ and $w_a \in W$ $\left(\begin{smallmatrix} i=1,\dots,n \\ a=1,\dots,m \end{smallmatrix}\right)$ $\forall i,j,a,b$ the matrix coefficients $\rho_{ij}^V(g)$ and ρ_{ab}^W are orthogonal (i.e. that $\langle \rho_{ij}^V(g), \rho_{ab}^W \rangle = 0$, where $\langle , \rangle$ is the inner product on $C^\infty(G, \mathbb{C})$ given by $\langle f_1, f_2 \rangle := \int_G f_1(g) \overline{f_2(g)}\, dg$.

(*)

(*)

(2) Let V be an irreducible representation of G and let $v_i \in V$ be an orthonormal basis with respect to the g-invariant inner-product. Then, the matrix coefficients $\rho_{ij}^V(g)$ are pairwise orthogonal, and each has norm squared $\frac{1}{\dim V}$: $\left(\rho_{ij}(g), \rho_{kl}^V \right) = \frac{\delta_{ik}\delta_{jl}}{\dim V}$

(3) and (4) aid in proving (1) and (2).

(3) Let V, W be non-isomorphic representations of G and f a linear map $V \to W$. Then, $\int_G gfg^{-1}\,dg = 0$.

(4) If V is an irreducible representation and f is a linear map $V \to V$, then $\int g f g^{-1} dg = \left(\frac{tr(f)}{\dim(V)}\right) id$.

Lemmas

$\boxed{\text{Pf (3+4)}}$ Let $\tilde{f} = \int_G gfg^{-1}dg$. $\tilde{f}$ commutes with the action of G: $h\tilde{f}h^{-1} = \int_G (hg) f (hg)^{-1} dg = \tilde{f}$.

Using Schur's Lemma, you get two possibilities ~~for~~ $\tilde{f}$.

Either $\tilde{f} = 0$ $(W \neq V)$ or $\tilde{f} = \lambda\, id$ $(W = V)$.

Since $tr(gfg^{-1}) = tr(f) \Longrightarrow tr\,\tilde{f} = tr\,f$.

Hence, $\lambda = \left(\frac{tr(f)}{\dim(V)}\right) id$

Big Picture: Irreducible representations enable us to construct orthonormal sets of functions on the group.

Problem: The sets of functions depend on your choice of basis.

Solution: We can leverage the character of a representation to circumvent this issue.

Def: A character of a representation V is the function on the group defined by

$$\chi_V(g) = tr_V\, \rho(g) = \sum \rho_{ii}^V(g).$$

(Clearly doesn't depend on your choice of basis.)

Lemma:

(1) Let $V = \mathbb{C}$ be the trivial representation. Then, $\chi_V = 1$.

(2) $\chi_{V \oplus W} = \chi_V + \chi_W$

(3) $\chi_{V \otimes W} = \chi_V \chi_W$

(4) $\chi_V(ghg^{-1}) = \chi_V(h)$

(5) Let V^* dual of a representation

The orthogonality relation for matrix coefficients immediately immediately implies the following theorem:

<u>Thm</u> (1) Let V, W be non-isomorphic complex irreducible representations of a compact real Lie group G. Then, the characters are orthogonal with respect to inner product
$\langle \chi_V, \chi_W \rangle = 0$.

(2) For any irreducible representation V,
$\langle \chi_V, \chi_V \rangle = 1$.

[Think Orthonormality]

If we denote, by $\hat{G}$, the set of irreducible isomorphism classes of irreducible representations of G, then the set $\{\chi_V, V \in \hat{G}\}$ is an orthonormal family of functions on G.

This is a really important realization as this gives way to many corrollaries, which will culminate in Peter-Weyl.

Corrollary Let V be a complex representation of a compact real Lie group G. Then,

(1) V is irreducible $\Leftrightarrow (\chi_V, \chi_V) = 1$.

(2) V can be uniquely in the form $V \cong \bigoplus n_i V_i$, where V_i are pairwise non-isomorphic irreducible representations, and the multiplicities n_i are given by $n_i = (\chi_V, \chi_{V_i})$.

But this is great! We now have a way to compute these multiplicities n_i. Unfortunately, this is only really ever usable for finite groups and some very special cases.

Returning to the topic of matrix coefficients of representations:

- Can we generalize (✳: Check Important Facts) without choosing a particular basis?
  - Yes.

Strategy:

Let $v \in V$, $v^* \in V^*$. We can define a function on the group $p_{v^*,v}(g)$ by

$p_{v^*,v}(g) = \langle v^*, p(g) v \rangle$.

Put simply, say $v = v_j$ and $v^* = v_i^*$.

$p_{v_i^*, v_j} = \langle v_i^*, p(g) v_j \rangle$ and we can recover the matrix coefficient $p_{ij}(g)$.

In other words, this means that

For any representation V, we have a map

$m: V^* \otimes V \to C^\infty(G, \mathbb{C})$

$v^* \otimes v \mapsto \langle v^*, p(g) v \rangle$

However, we should also note that $V^* \otimes V$ is a G-bimodule.

G-module: You have two commuting actions of G on your space $V^* \otimes V$.

If V is Unitary, then the inner product defines an inner product on V^*.

We can define an inner product on $V^* \otimes V$ by

$$(v_1^* \otimes \omega_1, v_2^* \otimes \omega_2) = \frac{1}{\dim V}(v_1^*, v_2^*)(\omega_1, \omega_2)$$

Thm Let $\hat{G}$ be the set of isomorphism classes of irreducible representations of G.
Define the map $m: \bigoplus_{V_i \in \hat{G}} V_i^* \otimes V_i \to C^\infty(G, \mathbb{C})$

by $m(v^* \otimes v)(g) = \langle v^*, \rho(g) v \rangle$

Then,

(1) The map m is a morphism of G-bimodules:

$$m((g v^*) \otimes v) = L_g(m(v^* \otimes v))$$
$$m(v^* \otimes g v) = R_g(m(v^* \otimes v))$$

where L_g and R_g are left and right actions of G on $C^\infty(G, \mathbb{C})$ respectively:

- $(L_g f)(h) = f(g^{-1} h)$
- $(R_g f)(h) = f(h g)$

(2) The map m preserves the inner product, if we define the inner product in $\bigoplus V_i^* \otimes V_i$ by

$$(v_1^* \otimes v_2, v_2^* \otimes w_2) = \frac{1}{\dim V} (v_1^*, v_2^*)(w_1, w_2)$$

and inner product in $C^\infty(G)$ by

$$(f_1, f_2) = \int_G f_1(g) \overline{f_2(g)} \, dg.$$

Pf | (1) Straightforward Computations

$(R_g m(v^* \otimes v))(h) = m(v^* \otimes v)(hg)$

$= \langle v^*, \rho(h)\rho(g)v \rangle = m(v^* \otimes gv)(h)$

$(L_g m(v^* \otimes v))(h) = m(v^* \otimes v)(g^{-1}h)$

$= \langle v^*, \rho(g^{-1})\rho(h)v \rangle = \langle gv^*, \rho(h)v \rangle$

$= m(gv^* \otimes v)(h)$

(2) Follows from previous theorem (☆) [Check Important facts]

Corollary: This map is injective.

Injective: By construction of mapping this is clear.

Note: Also surjective: Requires notions of completion. Every function on the group can be approximated by a linear combination of matrix coefficients.

Peter-Weyl Theorem

The map $m: \bigoplus_{V_i \in \hat{G}} V_i^* \otimes V_i \to C^\infty(G, \mathbb{C})$ [Same as before] gives an isomorphism.

$\hat{\bigoplus}_{V_i \in \hat{G}} V_i^* \otimes V_i \to L^2(G, dg)$

$\hat{\oplus}$ is the Hilbert Space Direct Sum

The completion of the algebraic direct sum with respect to the metric given by the inner product $(v_1^* \otimes w_1, v_2^* \otimes w_2) = \frac{1}{\dim V}(v_1^*, v_2^*)(w_1, w_2)$, and $L^2(G, dg)$ is the Hilbert Space of Complex Valued Square Integrable Functions on G with respect to the Haar measure, with the inner product $(f_1, f_2) = \int_G f_1(g) \overline{f_2(g)}\, dg$.

Corollary The set of characters $\{\chi_V, V \in \hat{G}\}$ is an orthonormal basis of the space of conjugation-invariant functions on G.

Ex: (Fourier Analysis)

Take $G = S^1 = \mathbb{R}/\mathbb{Z}$. The Haar measure on G is given by dx (shown previously) and the irreducible representations are parametrized by $\mathbb{Z}$:

$\forall k \in \mathbb{Z}$ we have a one-dimensional representation V_k with the action of S^1 given by $\rho(a) = e^{2\pi i k a}$.

The corresponding matrix coefficients are the same as the character and is given by $\chi_k(a) = e^{2\pi i k a}$.

The orthogonality relation of a previous theorem tells us $\int_0^1 e^{2\pi i k x}\overline{e^{2\pi i l x}}\,dx = \delta_{kl}$.

Intuition: Peter-Weyl is telling us that $\{e^{2\pi i k x}\}_{k\in\mathbb{Z}}$ form an orthonormal basis of $L^2(S^1, dx)$, which is the foundation of pretty much all of Fourier Analysis.

Every L^2 function on S^1 can be rewritten as $f(x) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi i k x}$ which converges in L^2 metric.

In short studying the structure of $L^2(G)$ gives us insight into and generalizes Harmonic Analysis / Fourier Analysis.