RIEMANN AND COMPLEX ALGEBRAIC GEOMETRY

CALEB JI

1. Bernhard Riemann

1.1. **Brief biography.** Riemann was born in 1826 to a humble family in Hanover, Germany. His father was a Lutheran pastor who taught him until he was ten. Riemann studied the classics, but he had a strong natural inclination toward mathematics which was clearly discernible in high school. Nevertheless he went to the University of Göttingen in 1846 to study theology. He could not stay away from mathematics though, and soon with his father's permission, he transferred to mathematics.

Göttingen was to soon to become the epicenter of mathematics in the 19th century, partially due to the influence of Riemann himself.[1] By the time Riemann entered as a student, Gauss had already served as a professor there for some forty years, and gave lectures there that Riemann attended. However, Gauss was somewhat notorious for considering lecturing a waste of time, and furthermore these were introductory courses which probably did not benefit too much from having Gauss be the lecturer. Riemann soon moved to Berlin in 1847, where he learned from Eisenstein and Dirichlet. In 1849 he returned to Göttingen to pursue his PhD, which he did under Gauss.

He impressed Gauss a lot on several occasions, which was not an easy thing to do, and was close with Dirichlet and Weber, whom he saw to be mentors. He stayed at Göttingen and eventually became a professor there. Although Riemann had difficulty lecturing at first, we cared deeply about it and improved with persistent practice. His works soon gave him a solid scientific reputation, though they were not always completely understood. He married Elise Koch in 1862, but his health was beginning to decline. He travelled to Italy for some time with his wife to try to improve it, and while he was happy there, his illness eventually took him away in 1866 at the age of 39. Riemann was always a humble and deeply religious man, and the freedom and creativity of his work reflected his pure heart.

1.2. **Works.** Riemann's thesis (1851) introduced Riemann surfaces and extended complex analysis to them. His 1857 paper on abelian functions studied continued their study; today we will focus on these two.

Looking at Riemann's papers, plenty of them are in applied math, science, or natural philosophy. The distinction between the flavor of these works and his work in pure mathematics was not so pronounced back then. Many of them were based on an incredibly deep and broad understanding of function theory in many guises.

Two papers which stand out, are his works on differential geometry and number theory. He prepared his paper on differential geometry because Gauss wanted to hear his thoughts on it. In it Riemann rethinks the notion of a metric, introducing Riemannian geometry. In number theory, Riemann studies the Riemann zeta function using complex analysis, considering its analytic continuation and proving its functional equation. He relates this to the distribution of the primes and poses the Riemann hypothesis. These two works can be viewed as the founding of Riemannian geometry and analytic number theory, two enormous fields of mathematics.

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[1] Gauss, Riemann, Klein, Dirichlet, Minkowski, Noether, and Hilbert are only some of the illustrious names associated to the mathematical school of Göttingen, before the Nazi party obliterated it with its anti-Semitism.

2. Complex analysis

The basics of complex analysis were worked out by the French mathematician Augustin-Louis Cauchy (1789 - 1856).

The notion of a holomorphic function is fundamental to complex analysis.

Definition 2.1. A function $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic at a point $z_0 \in \mathbb{C}$ if the limit

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists.

If this is the case, the derivative $f'(z_0)$ is taken to be this limit. This definition applies equally well to functions defined only an an open set $\Omega \to \mathbb{C}$, and as we will see later, extends to Riemann surfaces. We can decompose $f = u + iv$ into a combination of two real-valued functions. If f is holomorphic, then because h can approach 0 in any direction in the complex numbers, we can let it approach 0 both via the real direction and the purely imaginary direction. This leads to the Cauchy-Riemann equations.

Theorem 2.2. *A complex valued function $f\colon \Omega \to \mathbb{C}$ is holomorphic at z_0 if and only if the Cauchy-Riemann equations hold:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Indeed, the converse holds by noting that if $h \to 0$, then if $h = (h_1, h_2)$, then both h_1 and h_2 go to 0, allowing us to use the partial derivatives in x and y.

You may recall from analysis that a real function is called real analytic at a point x_0 if in some neighborhood of x_0, it equals a power series. This definition naturally extends to the notion of a complex analytic function. It turns out that being holomorphic and being complex analytic at a point are equivalent, thanks to some wonderful theorems by Cauchy.

Theorem 2.3 (Cauchy's theorem on a disk)**.** *If f is holomorphic on a disk, then for any closed contour C in that disk, we have*

$$\int_C f(z)\, dz = 0.$$

This was proven by Goursat in the case of a triangle. In general, one shows that there exists a primitive for f using the result for a triangle.[2] In fact Cauchy's theorem holds as long as f is holomorphic on a *simply connected* domain, which intuitively is one that has no holes. As a counter-example, consider $f(z) = \frac{1}{z}$, which is holomorphic on $\mathbb{C}\backslash\{0\}$. Integrating around the origin gives $2\pi i$. We will return to the notion of being simply connected later.

Theorem 2.4 (Cauchy's integral formula)**.** *Let f be holomorphic on a simply connected domain Ω. Then for $C \subset \Omega$ a closed contour around a point $z_0 \in \Omega$, we have*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz.$$

The proof is quite entertaining. One draws a contour keyhole using C but at two neighboring points on C, going into z_0 to cut z_0 out. The purpose of this is that this contour doesn't contain z_0, which is the only point on the domain where the function $\frac{f(z)}{z-z_0}$ is not holomorphic. So by the Cauchy integral formula, this part is 0. The two lines to z_0 are close enough that, passing to the limit, they cancel out. Finally, there is a small loop around z_0, on whose value f is arbitrarily close to $f(z_0)$. Then the fact that the integral of $\frac{1}{z}$ around the origin gives $2\pi i$ gives the desired result.

A key consequence of Cauchy's integral formula is that holomorphic functions are complex analytic, and in particular have infinitely many complex derivatives. Other applications include Liouville's theorem and the fundamental theorem of algebra, and analytic continuation.

[2] This may be reminiscent of Green's theorem (or Stokes' theorem), but if you want to prove Cauchy's theorem this way you may run into circular reasoning issues when looking at the conditions for Green's theorem.

Functions such as $\frac{1}{z}$ are meromorphic functions on the complex plane. Basically, these are holomorphic functions which are allowed certain singularities. These functions are very natural from the point of view of algebraic geometry, as they can be considered holomorphic functions onto the Riemann sphere that are not identically ∞. Indeed, we say that f has a pole at z_0 if $\frac{1}{f(z)}$ is holomorphic in a neighborhood of z_0, where we set its value at z_0 itself to be 0. In this case, around z_0 we can write

$$f(z) = \sum_{n=-N}^{\infty} a_n (z - z_0)^n.$$

Here it is important that N is finite. The coefficient a_1 is the **residue** of f at z_0.

Theorem 2.5 (Cauchy's residue theorem)**.** *Let f be meromorphic on a simply connected domain Ω. Then for $C \subset \Omega$ a closed contour on whose interior f is holomorphic except for a finite set of poles $z_1, \ldots, z_n$ at which f has residues $\mathrm{Res}_1(f), \ldots, \mathrm{Res}_n(f)$, we have*

$$\int_C f(z)\, dz = 2\pi i \sum_{i=1}^{n} \mathrm{Res}_i(f).$$

This can be proven in a similar way to Cauchy's integral formula.

3. Riemann's thesis

Riemann's doctoral thesis *Grundlagen für eine allgemeine Theorie der Funktionen einer veriänderlichen complexen Grösse* (Foundations for the theory of functions of a complex variable) represented a dramatic leap forward in the theory of complex variables, though it wasn't fully appreciated at the time. The main ideas he introduced are:

(1) Riemann surfaces as branched covers of $\mathbb{C}$ (or $\mathbb{P}^1_{\mathbb{C}}$), and the extension of complex analysis to them,
(2) The Dirichlet principle,
(3) Application to the Riemann mapping theorem.

We will mainly focus on (1) here. To extend Cauchy's work, it was natural for Riemann to consider "multi-valued functions" like $f(z) = z^{1/n}$, which – ignoring for now the questions of well-definedness – satisfy the Cauchy-Riemann equations. However, to have them make sense, their domain must be taken to be a surface, rather than the complex plane.

Indeed, given a function $w = f(z)$, where we view z as a local coordinate on a surface and w as a coordinate on $\mathbb{C}$, we can ask that this function verifies the Cauchy-Riemann equations. Then depending on the nature of f, the domain naturally becomes a surface with several sheets, one for each possible determination of the "multi-valued functions." While Riemann's predecessors simply viewed there as being cuts on the plane that would give separate functions, Riemann puts them together to form Riemann surfaces. Riemann doesn't give precise definitions – this will have to wait for Weyl in the early 20th century – and indeed, in many of his works he sometimes falls short of complete rigor, while still coming to essential great ideas.

Continuing on, Riemann studies the integrals $\int_C f(z)\, dz$ on Riemann surfaces where f is holomorphic and realizes they are not always zero because the surface S may not be simply connected. But by removing some curves C_i from S, the complement becomes simply connected. The integral of f about the C_i become the *periods* of the integral.

The Dirichlet principle is the idea that the solution to the Laplace equation is given by a the minimizer of a certain energy function. Riemann possibly named it after Dirichlet because he heard him use it in lectures, but it seems to not have originated with him. In the form Riemann used it it wasn't quite correct, and Weierstrass found a counterexample. Nevertheless, his particular use for his applications turned out to be eventually justified by Hilbert.

Riemann used the Dirichlet principle to prove the Riemann mapping theorem, which states that every open subset of the complex plane, other than its entirety, is *biholomorphic* (sometimes called *conformally equivalent*) with the open unit disk. In other words, there is a bijective holomorphism between the two. This eventually led to the uniformization theorem by Poincaré and Koebe.

Theorem 3.1 (Uniformization theorem, Poincaré (1907) and Koebe (1907))**.** *Every simply connected Riemann surface is biholomorphic with the Riemann sphere, the complex plane, or the open unit disk.*

Every Riemann surface has a Riemann surface as a universal covering, which by definition is simply connected. Thus every Riemann surface can be realized as a quotient of one of those three. For instance, elliptic curves are given by $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$. Note that the choice of τ matters, as different choices of τ may lead to topologically equivalent but algebraically/complex analytically distinct elliptic curve. The hyperbolic case (open unit disk) case is very interesting, and Poincaré explored it with his theory of Fuchsian functions.

4. The 1857 paper

In 1857, Riemann published *Theorie der Abel'schen Functionen* (Theory of abelian functions), an amazing work which poses new themes of research which were to permeate algebraic geometry for many decades to come. However, as one can tell by the title of the paper, his focus was on abelian functions. We may have occasion to return to this specific topic at a later date, but at the present time we will instead focus on other surrounding ideas and results of this paper most directly relevant to algebraic geometry, drawing from future developments as appropriate.

4.1. **Algebraicity of Riemann surfaces.** Approaching Riemann surfaces through complex analysis and function theory, Riemann apparently did not explicitly mention algebraic curves himself in his work. But the deep connection between his work and algebraic geometry lies in the fact that Riemann surfaces and complex algebraic curves are one and the same, a highly non-trivial fact that arose from Riemann's work. Indeed, in general there is no reason to expect a complex manifold to be an algebraic variety; i.e., given by solutions to polynomial equations. It is a beautiful result that in dimension 1, this turns out to be the case. We will just give a sketch of some of the ideas behind this.

The following fact is the key.

Theorem 4.1. *Let X be a compact Riemann surface. Then there exists a non-constant meromorphic function on X.*

As simple as this result sounds, it requires some hard analysis to prove. Granting it, there are two ways to conclude.

First, one can now consider the function field $K(X)$ of all meromorphic functions on X. For example, we have $K(\mathbb{P}^1)=\mathbb{C}(x)$. Indeed, if a meromorphic function has poles at some points p_i, we can multiply by powers of $z-p_i$ to make it holomorphic, and since it holomorphic at ∞ it has to be a polynomial; this means that meromorphic functions are rational functions. (The same is true for $\mathbb{P}^n$ but it is harder, using Chow's theorem.) Now our non-constant map $f\colon X\to\mathbb{P}^1$ induces an injection of function fields $K(\mathbb{P}^1)=\mathbb{C}(x)\hookrightarrow K(X)$. Now there is an equivalence of categories between:

(1) Fields of transcendence degree 1 over $\mathbb{C}$ and field inclusions,
(2) Smooth projective algebraic curves over $\mathbb{C}$ and dominant morphisms,
(3) Compact connected Riemann surfaces and non-constant holomorphic maps,

where the equivalence of the first two is a classical fact of algebraic geometry and the equivalence of the first and the third arises from using this f as we have done. Going between them, we get that compact connected Riemann surfaces are indeed algebraic. We will explain this idea more when we discuss Dedekind-Weber's algebraic approach to Riemann surfaces and number theory; it can be thought of as a prototype for the theory of schemes.

The second way is to show that there are enough of these meromorphic functions that we can use them to embed our Riemann surface into projective space. Then we use the algebraic dependence between these meromorphic functions (since they give a field of transcendence degree 1) to give algebraic relations on the coordinates to cut out our variety.

Example 4.2. It is helpful to consider the example of an elliptic curve. Recall that an elliptic curve can be described as

$$E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau),$$

where τ is an element of the upper half plane. Here $\mathbb{Z} + \mathbb{Z}\tau$ is an integer lattice in $\mathbb{C}$, which we will denote by Λ. We are going to construct meromorphic functions on this Riemann surface what will embed it into $\mathbb{P}^2$, with an algebraic relation between the functions.

Note that to construct a function on E, we need to construct a doubly periodic function on $\mathbb{C}$. Weierstrass did this with the $\wp$-function, defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

Its derivative is another meromorphic function, given by

$$\wp'(z) = -\sum_{\lambda \in \Lambda} \frac{2}{(z-\lambda)^3}.$$

These are algebraically related as expected, by the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(\Lambda)\wp(z) - g_3(\Lambda).$$

Here $g_2(\Lambda)$ and $g_3(\Lambda)$ are constants associated to the lattice; they are Eisenstein series.

Thus the map from E to $\mathbb{P}^2$ defined by $z \mapsto [\wp(z) : \wp'(z) : 1]$ for $z \notin \Lambda$ and mapping $z \in \Lambda$ to $[0 : 1 : 0]$ gives an isomorphism between E and the elliptic curve $y^2 z = 4x^3 - g_2(\Lambda)xz^2 - g_3(\Lambda)z^3$. The inverse is given by integrating a holomorphic form.

The generalization of this example to higher genus led to the study of abelian varieties. These ideas sprung from abelian integrals, and Riemann achieved fundamental results in this direction in his 1857 paper. (We may come back to this later.)

4.2. **Riemann-Roch.** Not content with showing that a meromorphic function simply exists, Riemann set out to determine when they exist given prescribed zeroes and poles, and the dimension of the vector space they span under such conditions. He achieved what is known as the Riemann inequality, while his student Roch proved the entire Riemann-Roch theorem. Extensions and generalizations of this theorem was to become a major theme in 20th century mathematics.

Let D be a divisor on a compact Riemann surface X. This means that D is a formal finite sum of points $\sum_i n_i P_i$; $n_i \in \mathbb{Z}, P_i \in X$. Let us set $\deg D$ to be the sum $\sum_i n_i$. We define $L(D)$ to be the set of meromorphic functions f such that f can only have poles at the P_i, and the order of the pole of f at P_i is at most P_i. For instance, if $D = -3P$, this corresponds to f with a zero of order at least 3 at P. Then $L(D)$ forms a $\mathbb{C}$-vector space, and we denote its dimension by $l(D)$.

Theorem 4.3 (Riemann's inequality). *On a compact Riemann surface with genus g, we have $l(D) \geq \deg D - g + 1$.*

This makes sense, as if you increase the degree, you should be allowing for more poles and thus more possible functions. For instance, on $\mathbb{P}^1$, which has $g = 0$, we have an equality. Indeed, if $D = \sum n_i P_i$, then a basis of $L(D)$ is given by $1, \ldots, (z - P_i)^{-1}, \ldots, (z - P_i)^{-n_i}, \ldots$. (We can do something similar if n_i is negative.)

Gustav Roch (1839 - 1866), Riemann's student whose life was cut short at 26 by tuberculosis, was able to make this into an equality by figuring out the difference between the two sides. To state his result, we must introduce the notion of a canonical divisor. Given a meromorphic function f, we can define $\operatorname{div} f$ to be the divisor determined by the sum of its zeroes minus its poles (with multiplicity). We can do the same thing for differential forms. It turns out that we can take div of any meromorphic 1-form to obtain a canonical divisor K.

Example 4.4. On $\mathbb{P}^1$, let us consider $\operatorname{div} dx$. It has no zeroes or poles on $\mathbb{C}$. At ∞, we can make the change of variables $x = y^{-1}$, so $dx = -y^{-2}\, dy$. This has a double pole at $y = 0$, or $x = \infty$, so $\operatorname{div}\ dx = -2\{\infty\}$.

For an elliptic curve, it turns out that 0 is a canonical divisor. In general, the degree of K is $2g - 2$.

Theorem 4.5 (Riemann-Roch). *On a compact Riemann surface with genus g and a canonical divisor K, we have*

$$l(D) - l(K - D) = \deg D - g + 1.$$

Let's check this for $\mathbb{P}^1$. This is simple because $l(D)$ only depends on its degree: it is $\deg D + 1$ if $\deg D \geq 0$ and is 0 otherwise. Since $\deg K = -2$ in this case, the result easily follows.

The quantity $l(K - D)$ has another natural interpretation, as the first Betti number corresponding to D (whereas $l(D)$ is the 0th). This follows from Serre duality; these developments eventually lead to the Grothendieck-Riemann-Roch theorem in the late 1950s.

4.3. **Riemann-Hurwitz and moduli of curves.** Let us begin with some topology. We have our notion of genus, which is the number of holes on a surface S. The genus is closely related to the Euler characteristic. For surfaces, the Euler characteristic $\chi(S)$ is equal to $V - E + F$ (vertices, edges, faces) of any triangulation, or any polygonal decomposition. There is a nice way to decompose a surface of genus g into a $4g$-sided polygon, extending the case of a (genus 1) torus. Actually, the 4g sides are identified in pairs so there are only $2g$ edges, and all the vertices are identified so there is only one face. Thus we get $\chi(S) = 2 - 2g$.

Let $f\colon X \to Y$ be a branched covering map of compact Riemann surfaces. This means that, outside of some finite set, it looks like a bunch of local isomorphisms. Over the finite set (the *ramification points*) where it does not, it will look like n points together, like z^n at 0. In fact, $z \mapsto z^n$ from $\mathbb{P}^1$ to $\mathbb{P}^1$ is a great example. The Riemann-Hurwitz formula relates the Euler characteristic of X and Y using the degree and ramification data of f.

First, consider the case when f is of degree d with no ramification points. Then given a triangulation of Y, its inverse image gives a triangulation of X, with every point – and thus edge and face – appearing d times. Thus in this case, we have $\chi(X) = d\chi(Y)$. The general case is not much more complicated.

Theorem 4.6 (Riemann-Hurwitz formula). *Let $f\colon X \to Y$ be a branched covering map of compact Riemann surfaces of degree d. Let the ramification points on Y be denoted P_i with ramificaiton degrees e_i. Then*

$$\chi(X) = d\chi(Y) - \sum_i (e_i - 1).$$

Proof. Begin with any triangulation of Y and refine it to include all the ramification points. Then the inverse image is a triangulation of X with everything appearing d times, except for the branch points which only appear once above the ramification points. Thus the triangulation of X gets $\sum_i (e_i - 1)$ less vertices than expected. This gives the desired formula. □

Since $\chi(X) = 2 - 2g_X$ and $\chi(Y) = 2 - 2g_Y$, we can also express this in terms of the genus. For example, we can use this to compute the genus of the source if we know the other data.

Example 4.7. Consider $\mathbb{P}^1 \xrightarrow{z^n} \mathbb{P}^1$. This map has degree n and its ramification points are 0 and ∞, both with ramification degree n. Thus the formula gives $2 - 2(0) = n(2 - 2(0)) - 2(n - 1)$, which is correct.

Example 4.8. Given an elliptic curve E with affine equation $y^2 = (x - a)(x - b)(x - c)$, the map to $\mathbb{P}^1$ given by sending $[x : y : z]$ to $[x : z]$ if $z \neq 0$ and sending $[0 : 1 : 0]$ to $[1 : 0]$ realizes E as a branched cover of $\mathbb{P}^1$ of degree 2, with 4 ramification points with ramification degree 2. Then Riemann-Hurwitz gives $2 - 2g_E = 2(2 - 2(0)) - 4 \Rightarrow g_E = 1$, as expected.

Riemann's name is attached to this formula because he used it in the case of $Y = \mathbb{P}^1$ to study the moduli space of curves. Given a fixed genus g, how can we describe the space of Riemann surfaces/smooth projective complex curves with genus g? While a fully satisfactory answer would have to wait till later, Riemann showed that for $g \geq 2$, the class of a curve of genus g depended on $3g-3$ parameters. As a warm-up, let's consider the cases $g=0$ and $g=1$. For $g=0$, by Riemann-Hurwitz any meromorphic function must have degree 1 with noramification points, and thus is an isomorphism. So $\mathbb{P}^1$ is the only curve with genus 0. When it comes to elliptic curves, the isomorphism class depends on the associated period lattice $\langle 1, \tau \rangle$. The value τ can take any value in the upper-half plane, and two values give the same elliptic curve if they are related by an element of $\mathrm{SL}(2, \mathbb{Z})$. This implies that elliptic curves depend on a single parameter.

The following argument is due to Riemann; it isn't exactly a theorem because the statement is a bit imprecise, but the idea is wonderful.

Theorem 4.9 (Riemann). *For $g \geq 2$, the isomorphism classes of compact Riemann surfaces of genus g depend on $3g-3$ parameters.*

Riemann's argument. Let X be a compact Riemann surface of genus g. By Riemann-Roch, for sufficiently large d any divisor D with degree d satisfies $l(D) = d - g + 1$ and $l(D - [p]) = d - g$ for all points $p \in X$.

Fixing such a d, we will study pairs (X, f) where X is a compact Riemann surface of genus g and f is a meromorphic function whose divisor of poles has degree d. On the one hand, by Riemann-Hurwitz the total ramification index of such a function has degree $2d+2g-2$. It turns out, using the fact that for $g \geq 2$, there are only finitely many automorphisms of X, that these choices of $2d + 2g - 2$ points on $\mathbb{P}^1$ determines all possible choices (X, f) to a constant factor. This implies that the dimension of the space of all such (X, f) (known today as a *Hurwitz space*) is $2d + 2g - 2$.

On the other hand, fixing X, we can determine how many parameters are used to determine such an f. First, by Riemann-Roch, once D itself is determined, there are $d - g + 1$ choices. Then there are d further choices to choose D, for a total of $2d - g + 1$ parameters.

Combining these, we see that X depends on $2d + 2g - 2 - (2d - g + 1) = 3g - 3$ parameters, as desired. □

A fully rigorous and satisfactory understanding of the answer to this question, that works not only over $\mathbb{C}$ but over all possible bases requires first of all a precise definition of a moduli space, and had to wait till the work of Deligne-Mumford on algebraic stacks.

Annotated bibliography

You can read Riemann's papers in their original German here: https://www.emis.de/classics/Riemann/. An English translation of them is given in the book *Collected Papers Bernhard Riemann* by Kendrick Press.

For more on Riemann's life, I recommend the final chapter of the Kendrick Press book.

For more on complex analysis, I recommend *Princeton Lectures in Analysis II, Complex Analysis* by Stein and Shakarchi (though the insistence on avoiding Riemann surfaces even when discussing elliptic functions is unfortunate).

For a great book on Riemann surfaces that develops many aspects of them in good detail, I recommend *Riemann Surfaces* by Donaldson.

For an introduction to algebraic geometry that includes the complex viewpoint described here, I highly recommend Matt Kerr's notes here: https://www.math.wustl.edu/~matkerr/AGbook.pdf (course website here: https://www.math.wustl.edu/~matkerr/436/index.html). Another excellent book on this topic is *Algebraic Geometry over the Complex Numbers* by Donu Arapura.