Weyl Integration Formula   Chang Yoon Seuk (uni: cs4056)

$\boxed{\text{Main Theorem}}$ for class function

Let G be a compact Lie gp; f be a class fnc on G; dg, dt be the normalized Haar measure on G and T, respectively. Then, we have the formula

$$\int_G f(g)\,dg = \frac{1}{|W|}\int_T f(t)\,\left|\det\left([Ad_{t^{-1}} - Id]|_{\mathfrak{p}}\right)\right|\,dt$$

Defn) Haar measure : Invariant measure defined on Borel-σ algebra.

What is Borel-σ algebra? σ-algebra generated by Borel sets. And, Borel-σ sets is the smallest collection of subsets of Top'l space X

i.e. The Borel set, denoted $\mathcal{B}(X) = \{$ opensets of X ; $\bigcup_{i=1} E_i$; $\bigcap_{i=1} E_i$, complements $\}$.
cnt'l union, cnt'le intersection.

e.g.) All open interval $(a,b) \in \mathcal{B}(\mathbb{R})$
" closed $[a,b] \in \mathcal{B}(\mathbb{R})$, Cantor set (uncountable, but measure zero)

Now, Haar measure on Lie gps? Haar integ'le fnc f on Lie gp G.

We know to integrate a fnc on mfd, we start with a fixed volume, and this requires mfd to be orientable. Clearly, any Lie gp is orien'le since $TG \simeq G \times \mathfrak{g}$, the tangent bundle is trivial; so, the volume form always exists in G.

Basic set up: Sps G is a cpt Lie gp, $T \subset G$ a maximal torus.

We know that the quotient G/T is homogenous G-manifold with tangent space

$T_{eT}(G/T) = \mathfrak{g}/\mathfrak{t} = \mathfrak{p}$. In this case, we fix an adjoint invariant inner product on $\mathfrak{g}$, and $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{t} \subset \mathfrak{g}$; this means

$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$; so, $\mathfrak{p} \subset \mathfrak{g}$. (i.e. we decompose Lie alg of G into two subspaces $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$

Explicitly, G-invariant inner product on $\mathfrak{g}$ is a bilinear form $\langle \cdot, \cdot \rangle$, satisfies

$$\langle Ad(g)x, Ad(g)y \rangle = \langle x, y \rangle \text{ for all } g \in G,\ x, y \in \mathfrak{g}.$$

, where $Ad(g)$ is just an adjoint representation $Ad(g)x = gxg^{-1}$

Why this is important? $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ be a well-defined decomposition.

$\mathfrak{t}, \mathfrak{p}$ be ortogonal, which is $\langle x, y \rangle = 0$ for $x \in \mathfrak{t}$, $y \in \mathfrak{p}$

$\Rightarrow$ i.e. Adjoint invariance ensures that $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ does depend on the choice of basis.

$\boxed{\text{Rmk}}$ What is homogenous G-manifold?

We know the following (applying notion of algebra) the orbit of G through mfd $m \in M$ is

$$G \cdot m = \{ g \cdot m \mid g \in G \} \subset M$$

, and the stabilizer (i.e. isotopic gp) of $m \in M$ is

$$G_m = \{ g \in G \mid g \cdot m = m \}.$$

Now from the smooth action $G \curvearrowright M$ is transitive if $G \cdot m = M$ for any $m \in M$.

meaning we have only 1 orbit!

Defn) A volume form ω on G is called left invariant if $L_g^* \omega = \omega \;\; \forall g \in G$.

Thm) A left invariant volume form exists on any Lie gp G ; unique upto mult've constant

proof. Take any basis of $T_e^* G$; to form non-zero elt, denoted $\omega_e \in \Lambda^n T_e^* G$.

Let $\omega_g = L_{g^{-1}}^* \omega_e$ be the n-form ω on G. Then we can varify 'left-translation', which is the left invariant since

$$(L_g^* \omega)_h = L_g^* \omega_{gh} = L_g^* L^*_{h^{-1}g^{-1}} \omega_e = (L_{h^{-1}g^{-1}} \circ L_g)^* \omega_e = L_{h^{-1}}^* \omega_e = \omega_h$$

Now, sps ω' be any left invariant volume form on G, $\dim \Lambda^n T_e G = 1$, there exists non-zero constant C, so that $\omega_e' = C \omega_e$. This follows from left-invariance for any g, $\omega_g' = L_{g^{-1}}^* \omega_e' = C L_{g^{-1}}^* \omega_e = C \omega_g$. So, we have shown uniqueness $\square$

Then, assume that $\omega > 0$, with respect to orienta'n of G, it gives a measure on G. which is $I(f) = \int_G f(g) \omega(g) = \int_G f\omega = \int_G L_h^* (f\omega) = \int_G (L_h^* f) \omega$

$$= I(L_h^* f) \text{ for any } h \in G.$$

In particular, for any Borel set $E \subset G$, with measure $m(E) = m(L_h E)$.
Such left invariant measure is called the 'left Haar measure'

<u>Quick explan'n.</u> Let G be locally cpt Lie gp (upto constant multiple).
A unique regular Borel measure μL is invariant under left transl'n
'Left transl'n invariance' : $\mu(X) = \mu(gX)$ for all measurable sets X, $g \in G$.
'Regularity' : $\mu(X) = \inf\{\mu(U) \mid U \supseteq X, U \text{ open}\} = \sup\{\mu(K) \mid K \subseteq X, K \text{ cpt}\}$.
Here, such a measure is called a 'left Haar measure'
'Property' : Any cpt set has finite measure & non-empty open set has measure > 0.
So, left-invariance of measure accounts to left-invariance of corresponding integral

$$\int_G f(hg)\, d\mu_L(g) = \int_G f(g)\, d\mu_L(g) \text{ for any Haar integ'le fnc } f \text{ on } G.$$

The Haar measure ω is called 'normalized' if $Vol(G) = \int_G \omega = 1$, denoted $\omega = dq$

And, since left-invariance means $d(hq) = dq$, we have

$$\int_G f(hq)\,dq = \int_G f(q)\,dq \quad \text{for any fixed } h \in G.$$

Then, we have the <u>Lemma</u>: there exists a normalized density $d(gT) = dq/dt$ on the quotient G/T, which invariant under G-action

Rmk) Density : Since Lie gp defines on a smooth structure, we may express the Haar measure in local cordinate as a density (i.e. smooth fnc) $\times$ standard Lebesgue measure.

i.e. the Haar measure $dq = \underbrace{f(q)}_{\text{density}} \cdot \underbrace{dz}_{\text{Lebesgue}}$

Ready to prove the <u>main thm</u> : the strategy $\begin{cases} \text{observe conjugacy classes} \\ \text{Regular points.} \end{cases}$ $\begin{cases} \text{use density \& Jacobian show} \\ \text{how the gp twists around max'l torus} \end{cases}$

Proof. Since we're dealing w/ class fncs, consider the map

$$\underset{\text{(not volume preserving)}}{\varphi} : G/T \times T \longrightarrow G$$
$$(gT, t) \longmapsto (gtg^{-1}) \quad \text{for any } g \in G,\ t \in T.$$

For a simpler case, we take the computa'n near identity : (eT, e).

So, fix g, t and construct fnc $\psi : G/T \times T \longrightarrow G$

$$(hT, s) \longmapsto (hts\,h^{-1}s^{-1})$$

Observe ψ in the composition map

$$\psi = R_{t^{-1}} \circ c(g^{-1}) \circ \varphi \circ (\check{L}_g \times L_t)$$

Let us inspect each part !

i) $R_{t^{-1}}$: Right trans'ln (i.e. multipl'n) by t^{-1}, sends our el't near t to Id.

$x \longmapsto x t^{-1}$

ii) $c(g^{-1})$: Jacobian determinant coming from left multil'n g^{-1}.

i.e. Conjugated by g^{-1} ; diff'l at pt gtg^{-1} ; volume preserving (by invariance of Haar measure under conjugation)

iii) φ : This is a natural conjugation map. sends the pt. :

$(ghT, ts) \longmapsto (gh)(ts)(gh)^{-1}$

iv) $\tilde{L}_g$: Left multipl'n by g on homogenous G/T

L_t : we already know this (left multipl'n by t on max'l torus T)

Together, $\tilde{L}_g \times L_t$ sends a pt $(hT, s) \longmapsto (ghT, s)$

Why? Recall the Haar measure! the density on G/T (given by dg/dt) is invariant under left G-action. (it is measure preserving).

Now, we already know dg & $d(gT)$ are G-invariant $\Rightarrow$ Corresponding Jacobian $=1$

Rmk) The Jacobian here shows how the volume at G splits in quotient space G/T & T

i.e. $|\det(\varphi)|$ transforms Haar measure on G to G/T & T

Then, applying i) ~ iv), $(d\psi)_{(eT,e)} = (dR_{t^{-1}})_t \circ (dc(g^{-1}))_{gtg^{-1}} \circ (d\varphi)_{(gT,t)} \circ d(\tilde{L}_g \times L_t)_{(eT,e)}$

~~Now~~ Since the Haar measure on (G, T) with induced density $d(gT)$ are invariant under left/right ~~trans~~ transl'ns and conjugation,

each of the map preserves the volume.

This means, their differ'ls have det = 1, which is

$$|\det(dR_{t^{-1}})| = |\det(dc(g^{-1}))| = |\det(d(\tilde{L}_g \times L_t))| = 1$$

Also, we notice $|\det(d\varphi)_{(gT,t)}| = |\det(d\psi)_{(eT,e)}| = 1$.

Compute $d\psi$ at (eT, e) given by (linearization first order approximation exponentials)

(idea: take $h = \exp(X)$, $X \in \mathfrak{p}$, $s = \exp(S)$, $S \in \mathfrak{t}$, then near (eT, e)

$\psi(\exp(X)T, \exp(S)) = \exp(X)\, t \exp(S) \exp(-X) t^{-1}$

for approximation take $\exp(X) \simeq I + X$, $\exp(S) \simeq I + S$, $\exp(-X) \simeq I - X$

to get $(Id + X)\, t\, (Id + S - X) t^{-1}$; so for $(X, S) \in \mathfrak{p} \times \mathfrak{t}$,

$d\psi(X, S)_{(eT, s)} = X + tSt^{-1} - tXt^{-1}$; since $Ad_t(h) = tht^{-1}$,

we can write $(d\psi)_{(eT, e)}(X, S) = (Id - Ad_t)(X) + Ad_t(S)$ for $X \in \mathfrak{p}, S \in \mathfrak{t}$

Clearly, this follows:

$$|\det(d\varphi)_{gT,t}| = |\det([Ad_{t^{-1}} - Id]|_{\mathfrak{p}}) \det Ad_t| = |\det(Ad_{t^{-1}} - Id)|_{\mathfrak{p}}|$$

Because, from our comput'n $d\psi$, observe from relative decompos'n $\mathfrak{p} \oplus \mathfrak{t}$,

the linear map would be $d\psi_{(eT, e)} \cong \begin{pmatrix} I - Ad_{t^{-1}}|_{\mathfrak{p}} & * \\ 0 & Ad_t|_{\mathfrak{t}} \end{pmatrix}$

↳ we see $|\det(d\varphi)_{(gT, t)}| = |\det[Ad_{t^{-1}} - Id]|_{\mathfrak{p}}| \cdot |\det Ad_t|$

since T is abelian and know that $|\det Ad_t| = 1$ $g \longmapsto |\det Ad_g|$

is Lie gp homomorphism from cpt G to $\mathbb{R}^+$.

Why? obvious. We already know that any Adjoint action is the linear map $Ad_g : \mathfrak{g} \to \mathfrak{g}$, $X \to gXg^{-1}$

; so, taking det gives a fnc $g \longmapsto |\det(Ad_g)|$

↳ Deduce into max'l torus $T \subset G$, s.t. $|\det Ad_t| = 1$ $\forall t \in T$

Another observation (Regular pts). Geometrically, any elt $t \in T$ is called 'regular' if its centralizer in G is exactly T. For a cpt Lie gp, the regular pts form a dense open subset (Irregular ones, det vanishes => has measure zero).

Obs 1) There exists dense open subsets $T^{reg} \subset T$ & $G^{reg} \subset G$ so that

$\det([Ad_{t^{-1}} - Id]|_p) \neq 0$ on T^{reg}, and φ is locally diffeomorphic

from $G/T \times T^{reg} \longrightarrow G^{reg}$

Obs 2) $\varphi(g_1T, t_1) = \varphi(g_2T, t_2) \iff t_1, t_2 \in T$ be conjugate in G

$\iff$ lie in the same W-orbit. φ is $|W|$-to-one corresponding map

from $G/T \times T^{reg} \longrightarrow G^{reg}$

Idea : For a fixed regular $t \in T^{reg}$; its conjugacy class of $G = \{gtg^{-1} : g \in G\}$

Since t be regular, every conjugacy of t arises exactly $|W|$ from different coset G/T represen've.

Final step summary : parametrize φ to ψ

$\downarrow$

Jacobian computation : $Jac(\varphi) = |\det[Ad_{t^{-1}} - Id]|_p|$

$\downarrow$

Regular pts (omit proof)

$\downarrow$

Use our lemma

$\downarrow$

Obtain formula for class fncs.

Using this fact, for any class fnc f, i.e. $f(gtg^{-1}) = f(t)$, we can write

$$\int_G f(g)\,dg = \int_{G/T\times T} f(\varphi(gT,t))\,d\mu$$, where $d\mu$ is the measure on $(G/T)\times T$, the Haar measure on G.

Since φ is not volume preserving, we use Jacobians of the change of variables.

We have $$\int_G f(g)\,dg = \frac{1}{|W|}\int_{G/T\times T} f(\varphi(gT,t))\,|\det(d\varphi)_{(gT,t)}|\,d(gT)\,dt$$

f is a class fnc & $\varphi(gT,t) = gtg^{-1}$, for $t\in T$, $f(\varphi(gT,t)) = f(t)$, then

$$\int_G f(g)\,dg = \frac{1}{|W|}\int_{G/T\times T^{reg}} f(t)\,|\det(Ad_{t^{-1}} - Id)|_{p}|\,d(gT)\,dt$$

, and since the density $d(gT)$ on G/T is normalized (= volume of 1)

, the integ'n over G/T just contributes a factor of 1, we obtain

$$\int_G f(g)\,dg = \frac{1}{|W|}\int_{T^{reg}} f(t)\,|\det(Ad_{t^{-1}} - Id)|_{p}|\,dt$$

Finally since the set of irregular elts has measure zero, we extend to all T

, and get the formula $$\int_G f(g)\,dg = \frac{1}{|W|}\int_T f(t)\,|\det([Ad_{t^{-1}} - Id]|_{p})|\,dt$$ $\square$

Cor) For any <u>cont's fnc</u> $f \in C(G)$, we define a $\check{f}$ on T by averaging over conjugacy $\check{f}(t) = \int_G f(gtg^{-1})\,dg$. This is W-invariant fnc on T, allows to identify as a class fnc on G : $\int_G f(g)\,dg = \int_G \check{f}(g)\,dg$.

So, by the Haar measure invarIance we get

$$\int_G f(g)\,dg = \frac{1}{|W|}\int_T \det([Ad_{t^{-1}} - Id]|_p)\underbrace{\left(\int_G f(gtg^{-1})dg\right)}_{=\check{f}(t)} dt$$

e.g.) $G = U(n)$ with max'l torus $T = \{\mathrm{diag}(e^{it_1}, \dots, e^{it_2})\ t_i \in [0, 2\pi)\}$

Again, dt be the normalized Haar measure on T. Followed from our formula, for each g is conjugate to diag matrix $t \in T$; so there exists $u \in U(n)$, s.t. $g = utu^{-1}$. We can define the map $\varphi : U(n)/T \times T \longrightarrow U(n)$

$$(uT, t) \longmapsto utu^{-1}$$

For a cls fnc f on $U(n)$, we can reparametrize up to Jacobian an $\frac{1}{|W|}$.

So, the formula tells us $\int_{U(n)} f(g)\,dg = \frac{1}{|W|}\int_T f(t)\,|\det[Ad_{t^{-1}} - Id]|_p|\,dt$

(again, dt is norm'd Haar & p is orthogonal complement of t i.e. $g = p \oplus t = u(n)$

$u(n)$ consists of Skew-Hermitian & p is subspace of off-diagonal matrices.

In this setting, $\det([Ad_{t^{-1}} - Id]|_p) = \prod_{j<k} |e^{it_j} - e^{it_k}|^2$ (product of eigen values over all off-diag matrices $\{j,k\}$ & $\{k,j\}$ for unordered → 2times

$$\det[Ad_{t^{-1}} - Id]|_p = \prod_{j \neq k}(e^{-it_j}e^{it_k} - 1)$$

$$= \prod_{j<k}(e^{it_j}e^{-it_k} - 1)(e^{it_k}e^{-it_j} - 1) = \prod_{j<k}(e^{it_j} - e^{it_k})(e^{-it_j} - e^{-it_k})$$

$$= \prod_{j<k}|e^{it_j} - e^{it_k}|^2$$

Background : basis E_{jk} $(j \neq k)$

$Ad_t(E_{jk}) = tE_{jk}t^{-1}$

E_{jk} be eigenvector of Ad_t

$\Rightarrow$ eigen value $= \lambda_{jk} = e^{i(t_j - t_k)} - 1$

, and since $U(n) \cong S_n \Rightarrow |W| = |n!|$

Hence, $\int_{U(n)} f(g)\,dg = \frac{1}{n!}\int_T f(t) \prod_{j<k}|e^{it_j} - e^{it_k}|^2\,dt$