Lie Correspondence

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Fundamental theorems of Lie theory (3 main results)

Initially, we have seen that for a Lie group G, with Lie algebra $\mathfrak{g}$, the exponential map $\exp : \mathfrak{g} \longrightarrow G$ is a local homeomorphism. So Lie algebras are locally determined by Lie groups.

The main theorem we want to show is:

Theorem A) The functor $\text{Lie} : G \longrightarrow \text{Lie}(G)$ is to be an equivalence of categories.

i.e. The assignment $\{\text{simply-connected Lie groups}\} \xrightarrow{\text{Lie}} \{\text{Lie algebras}\}$ is functorial.

Theorem B) For any Lie group G (real or complex), there is a bijection between connected Lie subgroups $H < G$ and Lie subalgebras $\mathfrak{h} < \mathfrak{g}$, given by the Lie functor.

Observation of Thm A)

Definition [Categories]. Informally, a mathematical structure consist of objects and maps or morphisms between objects. Two main axioms: associativity and identity

Definition [functor]. Let C, D be two categories. A functor $F : C \longrightarrow D$ is an assignment that

1) Assigns $c \longmapsto F(c)$ for each $c \in C$, 2) To each morphism $f : c_1 \longrightarrow c_2$ in C, a morphism $F(f) : F(c_1) \to F(c_2)$

Two rules: 1) Identity $F(id_c) = id_{F(c)}$, 2) Composition $F(g \circ f) = F(g) \circ F(f)$.

Definition. A functor $F : C \longrightarrow D$ is an equivalence of categories if

1) F is fully faithful. This means, for every pair of objects c_1, c_2 in C, the map $F : \text{Hom}_C(c_1, c_2) \longrightarrow \text{Hom}_D(F(c_1), F(c_2))$ is a bijection.

i.e. F induces a one-to-one correspondence between morphisms in C and D.

2) F is essentially surjective on objects. For every object $d \in D$, there exists $c \in C$, such that d is isomorphic to $F(c)$

Definition [simply-connected]: $\begin{cases} \text{Path-connected} \\ \text{Trivial fundamental group} \end{cases}$

Definition [Path-connected]. A topological space T is path-connected if, for every points $x, y \in T$, there exists a continuous map $\gamma: [0,1] \longrightarrow T$, such that $\gamma(0) = x$ and $\gamma(1) = y$.

Definition [Homotopy of paths]. Let f and f' be continuous maps of the space X into the space Y, we say f is homotopic to f' if there is a continuous map $F: X \times I \longrightarrow Y$, such that $F(x,0) = f(x)$ and $F(x,1) = f'(x)$ for each x. ($I = [0,1]$).

Equivalently, homotopy of loops $H: I \times I \longrightarrow X$, with two loops ℓ_1, ℓ_2 at $x_0 \in X$ defined by $\begin{cases} H(s,0) = \ell_1(s), \text{ for all } s \in I \\ H(s,1) = \ell_2(s), \text{ for all } s \in I \\ H(0,t) = H(1,t) = x_0 \text{ for any } t \in I \end{cases}$, we say $\ell_1 \simeq \ell_2$.

i.e. Homotopy of loops is that one can be constantly deformed into the other while keeping fixed based point x_0.

Definition [Fundamental group] $\pi_1(X, x_0)$: Set of path homotopy classes of loops based at x_0, with the group operation $*$. The group operation is given by concatenation of loops: $[\ell_1] \cdot [\ell_2] = [\ell_1 * \ell_2]$, where $(\ell_1 * \ell_2)(t) = \begin{cases} \ell_1(2t), & 0 \le t \le \frac{1}{2} \\ \ell_2(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$

'Trivial fundamental group' is then $\pi_1(X, x_0) \simeq \{e\}$, meaning, every loop at x_0 is homotopically equivalent to the constant loop.

e.g. $\pi_1(\text{contractible } X) = 0$. We know that a space X is contractible if the identity map $id_X: X \longrightarrow X$ is homotopic to a constant map C_{x_0}, such that $H: X \times [0,1] \longrightarrow X$, for all $x \in X$ defined by $H(x,0) = x$ and $H(x,1) = x_0$. So, we can see $\pi_1(\mathbb{R}^n) = 0$

e.g.) $\pi_1(S^1) \cong \mathbb{Z}$ (Idea from covering spaces $p: \mathbb{R} \longrightarrow S^1$, $p(t) = e^{2\pi i t}$, which is universal cover of S^1. Any loop in S^1 can be lifted to a path in $\mathbb{R}$, and we consider integer as loops' winding number. i.e. $n > 0$: ℓ goes counterclockwise n times

$n < 0$: // clockwise $|n|$ times

$n = 0$: contracted to a base point.

Rmk. We can also view π_1 as a functor between categories $\pi_1: \text{Top}'_* \longrightarrow \text{Group}$

Top_* is the category of topological space with pairs (X, x_0). We have action on objects and action on morphisms $f: (X, x_0) \longrightarrow (Y, y_0)$, then $\pi_1(f): \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$, defined by composition of loop ℓ in X, $\pi_1(f)[\ell] = [f \circ \ell]$. So, π_1 is a well-defined functor.

Rmk. Lie group functor $\mathrm{Lie} : \{\text{Lie groups}\} \longrightarrow \{\text{Lie algebras}\}$ defined by
$\{f : G \to H\} \longmapsto \{f_* : T_eG \longrightarrow T_eH\}$.

Now, we have shown definitions contained in Theorem A, then we split this into

<u>Theorem A_1</u>) If G_1, G_2 be Lie groups and G_1 is connected and simply-connected, then $\mathrm{Hom}(\mathfrak{g}_1, \mathfrak{g}_2) = \mathrm{Hom}(G_1, G_2)$, where $\mathfrak{g}_1, \mathfrak{g}_2$ are Lie algebras of G_1, G_2 respectively.

<u>Theorem A_2 [Lie's third theorem]</u>) Any finite-dimensional (real or complex) Lie algebra is isomorphic to a Lie algebra of Lie group.

To prove A, we first need to prove thm B. We have two approaches.

Proof 1) Using Baker-Campbell-Hausdorff (BCH-formula)

We need to construct a Lie subgroup $H < G$, for every Lie algebra $\mathfrak{h} \subset \mathfrak{g}$.
Assume that $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra. We wish to produce a connected Lie subgroup $H < G$, with $\mathrm{Lie}(H) = \mathfrak{h}$. The key step is to show $\exp(\mathfrak{h}) \subset G$ generates a subgroup, and that subgroup is locally closed under multiplication by BCH.
Now, for any $x, y \in \mathfrak{h} \subset \mathfrak{g}$, and by BCH, we consider $\exp(x)\exp(y) \in G$. This is because we already know $\exp(x)\exp(y) = \exp(\mathrm{BCH}(x,y))$, where
$\mathrm{BCH}(x,y) = x + y + \frac{1}{2}[x,y] + \frac{1}{12}[x,[x,y]] - \frac{1}{12}[y,[x,y]], \ldots$ is an infinite series of nested commutators. Since $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, all those nested commutators remain in $\mathfrak{h}$. Hence, $\mathrm{BCH}(x,y) \in \mathfrak{h}$, and thus, $\exp(x)\exp(y) = \exp(\mathrm{BCH}(x,y)) \in \mathfrak{h}$.
Next, for the connected Lie algebra of H, let H° be connected subgroup generated by $\exp(\mathfrak{h})$. i.e. the identity component generated by $\exp(\mathfrak{h})$. But, since BCH remains in $\mathfrak{h}$, which implies $T_e(H^\circ) \supset \mathfrak{h}$, we easily see $\mathrm{Lie}(H^\circ) = \mathfrak{h}$.
To sum, we have shown Lie subgroup $\xrightarrow{\mathrm{Lie}}$ Lie subalgebra and Lie subalgebra $\xrightarrow{\mathrm{Lie}}$ Lie subgroup

$H \longmapsto \mathrm{Lie}(H$ $\qquad\qquad$ $\mathfrak{h} \longmapsto \langle \exp(\mathfrak{h}) \rangle^\circ = H$

Hence, there is a bijection between connected Lie subgroups and Lie subalgebras given by the Lie functor □

Proof 2) Requires some background in differential geometry.

[Sketch of the proof] We will introduce theorems by Frobenius and the following lemma.

Theorem [Frobenius integrability criterion]

A distribution $\mathcal{D}$ on a smooth manifold M is completely integrable if and only if for any two vector fields $u, v \in \mathcal{D}$, one has $[u,v] \in \mathcal{D}$.

Theorem. Let $\mathcal{D}$ be a completely integrable distribution on manifold M (smooth). Then, every point $p \in M$, there exists a unique connected immersed integral in submanifold $N \subset M$ of $D \ni p$, and it is maximal.

Remark) An immersed: the submanifold $N \subset M$ is immersed integral manifold for V if, for every $p \in N$, the immerse of the map $dI_N : T_pN \longrightarrow T_pM$ is V_p.

Remark) Integrable distribution: A n-dimensional distribution on a manifold M is a n-dimensional subbundle $D \subset TM$. Formally, for all $p \in M$, we have n-dimensional subspace $D_p \subset T_pM$ (smoothly depends on P). We can see this in ODE, which is a well-known notion of directional field. In differential geometry, for a vector field v, we write $v \in D$, if for every point p, we have $v(p) \in D_p$. A straightforward generalization of integral curve is then the integral manifold for a distribution D is a n-dimensional submanifold $X \subset M$, such that for every $p \in X$, we have $T_pX = D_p$.

Now, for a given Lie group G with Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, corresponds to $H < G$, we notice that if H exists, then at every point $p \in H$, $T_pH = (T_eH)p = \mathfrak{h} \cdot P$. So, our H H will be an integral manifold of the distribution D^{h}, where $D^{h}_{p} = \mathfrak{h} \cdot P$. Then we have the following lemma to construct H.

Lemma For every point $q \in G$, there is locally an integral manifold of the distribution D^{h} containing q, denoted $H^{0} \cdot q$, where $H^{0} = \exp \mathcal{U}$ for some neighborhood $\mathcal{U}$ of 0 in $\mathfrak{h}$.

Using this lemma, we can prove our theorem B.

Now, we will show A_1. By the result of thm B, we already know morphisms of Lie groups define morphisms of Lie algebras. And, for a connected Lie group G_1, the map $\mathrm{Hom}(G_1, G_2) \longrightarrow \mathrm{Hom}(\mathfrak{g}_1, \mathfrak{g}_2)$ is injective. So, we are only left to show surjectivity. This means, every morphism of Lie algebras $f: \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ can be lifted to a morphism of Lie groups $\varphi: G_1 \longrightarrow G_2$ with $\varphi_* = f$.

We define $G = G_1 \times G_2$, then Lie algebra of G is now $\mathfrak{g}_1 \times \mathfrak{g}_2$.

Let $\mathfrak{h} = \{x, f(x) \mid x \in \mathfrak{g}_1\} \subset \mathfrak{g}$, which is subalgebra, and obviously it's a subspace.

So, we can express this as a commutator:

$$[(x, f(x)), (y, f(y))] = ([x,y], [f(x), f(y)]) = ([x,y], f([x,y]))$$

Then by B), there is a corresponding connected Lie supgroup $H \hookrightarrow G_1 \times G_2$.

We can compose this embedding by projection $p: G_1 \times G_2 \longrightarrow G_1$, and we get a morphism $\pi: H \longrightarrow G_1$ with $\pi_*: \mathfrak{h} = \mathrm{Lie}(H) \longrightarrow \mathfrak{g}_1$. (view π as a covering map)

In fact, since G_1 is simply-connected, H must be connected, π must be isomorphism.

i.e. we also have $\pi^{-1}: G_1 \longrightarrow H$. Now, we construct a map $\varphi: G_1 \longrightarrow G_2$ as a composition

$$G_1 \xrightarrow{\pi^{-1}} H \hookrightarrow G_1 \times G_2 \longrightarrow G_2$$; clearly, this is a morphism of Lie groups.

Also, $\varphi_*: \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ is a composition.

$$x \longmapsto (x, f(x)) \longmapsto f(x)$$

Hence, we have lifted f to a morphism of Lie groups, so the functor Lie is surjective $\square$

Proof of A_2: Lie's third theorem [Ref. Terrance Tao on Ado's theorem]

In the special case of using adjoint representation $\mathrm{Ad}: \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ on itself defined by the action $X: Y \longmapsto [X, Y]$; the Jacobi identity ensures that Ad is a representation of $\mathfrak{g}$.

The kernel is the centre $Z(\mathfrak{g}) = \{X \in \mathfrak{g} : [X, Y] = 0, \text{for all } Y \in \mathfrak{g}\}$. In this case, $\mathfrak{g}$ is semi-simple.

Now, Ado's theorem says, assume that $\mathfrak{g}$ is a finite dimensional Lie algebra over a field k with characteristic 0, then there is a faithful finite dimensional representation of $\mathfrak{g}$, which is an embedding $\mathfrak{g} \hookrightarrow \mathfrak{gl}(N, k)$ for some N.

Cartan also gave different approach

Proof of A_2 left as a challenge!