HISTORICAL OVERVIEW OF LIE THEORY

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Abstract. We give a brief historical overview of Lie theory beginning with the work of Lie. Along the way, we introduce the fundamental definitions of Lie groups and Lie algebras and a few other essential notions. All the material here is well-known, with the historical parts coming from [2, 3, 1].

1. Sophus Lie

1.1. **In lieu of a biography.** Sophus Lie (1842 – 1899) was a profoundly original Norwegian mathematician. He studied science at the University of Christiana (now Oslo), but didn't show any particular affinity to mathematics at the time. Instead, he considered being an astronomer. This naturally led him to read the work of Plücker, learning the cutting-edge geometry of the time. His interest soon shifted to mathematics, and he went to Berlin to learn from the top mathematicians of the day. While he met Kummer, Kronecker, and Weierstrass there, their style of mathematics didn't attract him. Instead, he met another visitor, Felix Klein, with whom he shared a close affinity. Despite different personalities and backgrounds, they were both geometers at heart.

The initial subject which united Lie and Klein was the study of line complexes. Plücker had initiated a theory of geometry in which lines, rather than points, were the basic building blocks. This point of view is taken for granted today in projective geometry, and Plücker's viewpoint his commemorated through Plücker coordiantes on Grassmannians.

From here, Lie's personal story takes many twists and turns, from being mistaken for a spy, taking opium, and the breakdown of his relationship with Klein, his student Engel, and others. Beginning his mathematical life as an outsider, his tremendous creativity and forceful character carried him to grand new heights in mathematics as well as acerbic conflicts. Details may be found in , and is good material for further discussions. For now, we will focus on his mathematics.

1.2. **Transformation groups and Lie's theorems.** In Lie's time, the notion of a manifold had not yet been fully formalized. This did not prevent Lie from proving many results of relevance to manifolds. Indeed, the notion of a Lie group is that of a manifold endowed with a compatible group structure, but Lie himself focused on the group theory and the local aspects.

Definition 1.1 (group)**.** *A group G is a set G endowed with a binary operation $* : G \times G \to G$ satisfying the following properties:*

- *(Identity) There exists an identity element $e \in G$ such that $e * g = g * e = g$ for all $g \in G$*
- *(Inverses) For all $g \in G$, there exists an inverse element h such that $gh = hg = e$.*
- *(Associativity) $a * (b * c) = (a * b) * c$*

You are familiar with many examples of groups: $(\mathbb{Z}, +), (\mathbb{R}, +), (\mathbb{Z}/n\mathbb{Z}, +), ((\mathbb{Z}/n\mathbb{Z})^*, \times), S_n$, but the ones of most interest in Lie theory are those such as $(\mathrm{GL}(n, \mathbb{R}), \times)$.

Groups like $(\mathrm{GL}(n, \mathbb{R}), \times)$ are interesting first of all because they have a geometric structure ($(\mathbb{R}, +)$ does as well) via viewing the matrix entries as coordiantes in space, which makes them into a Lie group. (We will discuss these in more detail later.) From another perspective, they

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act on spaces ($\mathbb{R}^n$ in this case), and thus can be viewed as transformation groups. Lie took this viewpoint and developed the theory of transformation groups in great detail. He related them to infinitesimal transformation groups, which are known today as Lie algebras. The relation between them can be summarized by Lie's three theorems.

Theorem 1.2 (Lie's first theorem)**.** *If two Lie groups are locally isomorphic, then so are their Lie algebras.*

Theorem 1.3 (Lie's second theorem)**.** *If G and H are Lie groups with G simply connected and there exists an isomorphism f between their Lie algebras, then f lifts to an isomorphism between G and H.*

Theorem 1.4 (Lie's third theorem)**.** *Every finite-dimensional Lie algebra is the Lie algebra of a Lie group.*

Lie didn't state and prove these theorems exactly in this form; e.g. the third one in the form stated above is due to Cartan, and is thus sometimes called the Cartan-Lie theorem. These results hold for both real and complex Lie groups.

Aside from Lie theory itself, Lie authored thousands of pages of quality research on topics such as differential equations, contact geometry, minimal surfaces, and the Erlangen program. However, his personal relations soured, even with Engel, toward whom he had been a close mentor. Lie's student Engel contributed to Lie's work on transformation groups and was responsible for authoring a large part of their joint work. Engel also began sharing Lie's ideas with one Wilhelm Killing, which ended up negatively affecting his relationship with his former mentor. However, this led to incredible and unexpected new mathematics by Killing which gave new life to Lie theory.

2. Wilhelm Killing

2.1. **Fast facts.** Wilhlem Killing (1847 – 1923) was a German mathematician who studied at Münster and then at Berlin. He did his thesis under Weierstrass and spent the next ten years in Braunsberg. Though mathematically isolated there, Killing produced remarkable and original work, reaching the study of Lie algebras independently of Lie. He eventually came into contact with Engel and Lie, and while he continued a productive correspondence with Engel, his personality clashed with Lie. Nevertheless, he is responsible for one of the most significant results in all of mathematics: the classification of semisimple Lie algebras. Killing was a German patriot and a lover of tradition, philosophy, and the classics. His later life was filled with tragedy, from the collapse of German society after WWI to the loss of his children.

2.2. **Lie algebras.** Lie himself wasn't much concerned with the study of Lie algebras themselves, but Killing was. Abstractly, a Lie algebra is defined as follows.

Definition 2.1. *A Lie algebra $\mathfrak{g}$ is a vector space over a field equipped with an alternating bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, denoted by brackets, satisfying the Jacobi identity:*

$$[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0.$$

The basic example of a Lie algebra is $\mathfrak{gl}(n,\mathbb{R})$, with Lie bracket given by $[x,y] = xy - yx$. While Lie algebras can be defined over arbitrary fields, for now we will stick with the real or complex case. Lie algebras come from Lie groups, as stated in Lie's three theorems, but since we haven't even formally defined manifolds yet, we will postpone discussion of this to the next section. Instead, let us explain the notion of simple and semisimple Lie algebras, which Killing's result is about.

Definition 2.2. *A simple Lie algebra is a nonabelian Lie algebra with no non-zero proper ideals.*

By requiring the Lie algebra to be nonabelian, we are excluding Lie algebras where the Lie bracket is always 0. Indeed, since the Lie bracket is alternating, if $[a, b] = [b, a]$ then it is also equal to $-[b, a]$, and provided we are not in characteristic 2 this implies $[a, b] = 0$. For the definition of ideal, recall that an ideal I in a commutative ring R is an additive subgroup such that $rI \subset I$ for all $r \in R$. In the context of a Lie algebra $\mathfrak{g}$, this condition becomes $[g, x] \in I$ for all $g \in \mathfrak{g}, x \in I$.

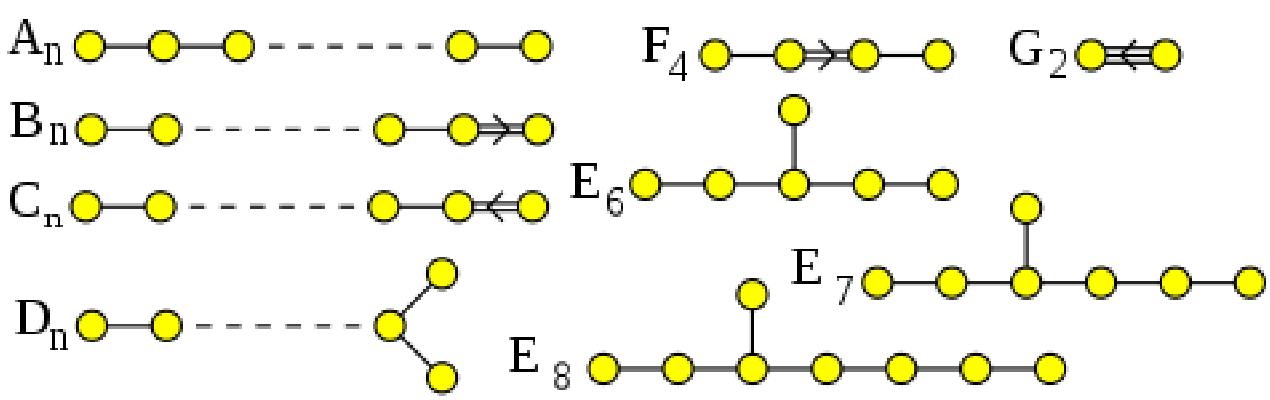
There are several equivalent ways to define semisimple Lie algebras.

Definition 2.3. *A Lie algebra is semisimple if it satisfies one of the equivalent (over characteristic 0) conditions:*

- *It is a direct sum of simple Lie algebras.*
- *Its Killing form is non-degenerate.*
- *Its radical is 0.*

Giving the definition of the Killing form and radical now may make them seem uninspired or arbitrary, which is not at all the case. We will return to them later in the semester when we have more context. For now, we will simply state the classification.

2.3. **Classification of semisimple Lie algebras.** Killing's classification of semisimple Lie algebras is stated through root systems. A root system is given by a certain finite graph which represents geometrical properties which we will explain in detail when we get to this point in the seminar. For now, we will just draw them. The root systems below correspond to the simple Lie algebras, while semisimple Lie algebras are given by unions of them.



3. Élie Cartan and Hermann Weyl

3.1. **Brief historical background.** Élie Cartan (1869 – 1951) and Hermann Weyl (1885 – 1955) were legendary mathematicians who contributed greatly to the shaping of mathematics in the early twentieth century (one could also include Élie's son, Henri Cartan, in this statement). While Cartan was began his doctorate at the École Normale Supérieure, he came across Lie and Killing's work and set about to complete it. Indeed, Cartan reworked much of Killing's work on Lie algebras, introducing his own ideas, and perfected their classification in his thesis. This was just the beginning of his work in Lie groups, which Cartan developed to a new level of maturity. As would characterize 20th century mathematics to come, Cartan codified new structure and built new theory that would support research for years to come. This was true not only in Lie theory but also differential geometry and surrounding fields.

Cartan was aided in his endeavors by the incredible work of Hermann Weyl. One of Weyl's many key contributions was the precise definition of a manifold. While mathematicians had implicitly worked with manifolds for some time, laying the foundations in this way allowed them to go much further in the years to come. Weyl would go on to develop Lie theory further through his work in representation theory, with the Weyl integration formula and the Weyl character formula being two important theorems which bear his name. Weyl was a universal mathematician, akin to Hilbert before him. Along with Cartan he also greatly contributed to mathematical physics and relativity, which use their earlier work on differential geometry and Lie groups.

3.2. **Manifolds and Lie groups.**

Definition 3.1. *A (topological) manifold is a Hausdorff topological space that is locally homeomorphic to $\mathbb{R}^n$ around every point.*

There are variants to this definition; e.g. second-countability of the topological space is assumed, but these considerations will not be important for us. Sometimes people want to differentiate between topological, differentiable, or smooth, or analytic manifolds. In such cases it is better to take the following definition: A manifold consists of an open covering $\{U_i\}$ and an atlas of maps $u_i\colon U_i \to \mathbb{R}^n$, which are each homeomorphisms onto $\mathbb{R}^n$, such that the transition maps $u_i \circ u_j^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ are continuous/differentiable/smooth/analytic. Two atlases are equivalent if their union is an atlas. Not all atlases are equivalent on a given manifold; this leades to the highly interesting phenomenon of multiple smooth structures on a manifold.

In this class, we will be primarily interested in smooth manifolds.

Definition 3.2. *Given smooth manifolds M, N with atlases $\{U_i\}, \{V_j\}$, a smooth map $f\colon M \to N$ is one such that the induced transition maps $v_j \circ f \circ u_i^{-1}$ are smooth.*

Now we come to the definition of a Lie group.

Definition 3.3. *A Lie group G is a smooth manifold with a group structure given by $m\colon G \times G \to G$ (multiplication) and $i\colon G \to G$ (inversion) such that m and i are smooth.*

A homomorphism of Lie groups is given by a map $f\colon G \to H$ which is simultaneously a smooth map and a group homomorphism.

An important point to mention is that complex manifolds and complex Lie groups also exist. These are important even for the study of Lie theory over $\mathbb{R}$, but for the time being we will focus on the real case.

Definition 3.4. *The tangent bundle TM of a manifold M of dimension n is a manifold of dimension $2n$ constructed as follows.*

Given an atlas $\phi_i\colon U_i \to \mathbb{R}^n$, we take TM to be the union of all $U_i \times \mathbb{R}^n$ under the equivalence relation $(x, v) \sim (x, \phi_j\phi_i^{-1}v)$ where $x \in U_i \cap U_v$.

The tangent bundle naturally comes with a projection $TM \to M$ with each fiber simply being $\mathbb{R}^n$. The tangent space T_xM for $x \in M$ is this fiber; these vector spaces fit together to form the tangent bundle.

There are other ways to think about tangent spaces. For example, one definition of the tangent space T_xM is an equivalence class of all smooth curves parameterized by $p : [-1, 1] \to M$ with $p(0) = x$, with the equivalence class being $p \sim q$ if $p'(0) = q'(0)$. This approach has the advantage of being very intuitive.

3.3. **Examples of Lie groups.**

Example 3.5. (1) $\mathbb{R}^n$ is a Lie group under addition.

(2) The torus $\mathbb{T}^n\colon (S^1)^n \cong (\mathbb{R}/\mathbb{Z})^n$ is a Lie group under addition.

(3) The general linear group $GL(n, \mathbb{R})$ of $n \times n$ invertible matrices is a Lie group under multiplication.
(4) The orthogonal group $O(n, \mathbb{R})$ consists of matirces A with $AA^T = I_n$. The special orthogonal group $SO(n, \mathbb{R})$ is the subgroup of $O(n, \mathbb{R})$ with determinant equal to 1.
(5) The symplectic group $Sp(2n, \mathbb{R})$ consists of matrices A that preserve an alternating bilinear form. That is, if $\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, then we take A such that $A^T M A = M$.

Let us motivate the definition of the symplectic group a bit by reinterpreting the orthogonal group with some linear algebra. We can think of the orthogonal group as the matrices which preserve a certain symmetric bilinear form B, namely the one given by the identity. Indeed, a form is given by $B(v, w) = w^T B v$, and changing basis gives $B' = A^T B A$.

Let us now explain one non-trivial result, showing that the group structure in a Lie group does indeed have concrete (and non-obvious) geometric consequences. A manifold is called *parallelizable* if its tangent bundle ist trivial, i.e. isomorphic to $M \times \mathbb{R}^n$. This is equivalent to there being a set of n smooth vector fields; i.e., smooth sections of $TM \to M$, which generate $T_x M$ at each $x \in M$. For example, R^n itself is parallelizable, and so is S^1.

It turns out that any Lie group is parallelizable. Indeed, if $\mathfrak{g}$ is the tangent space of a Lie group G at the identity, we have an isomorphism $G \times \mathfrak{g} \cong TG$ given by $(g, v) \mapsto (g, L_{g*} v$. Indeed, the fact that L_g (left multiplication by g) is smooth and a homeomorphism implies that this is indeed an isomorphism. Alternatively, one can think of left multiplication by g as moving a basis of $\mathfrak{g}$ around G to form the desired vector fields.

Which surfaces admit a Lie group structure? We have already seen that the torus (genus 1) does, but the sphere S^2 does not. Indeed, the hairy ball theorem implies that S^2 is not parallelizable. What about higher genus?

Theorem 3.6 (Poincaré-Hopf)**.** *A vector field v on a compact differentiable manifold M with isolated zeroes satisfies $\sum_{x \in M} \mathrm{ind}_x(v) = \chi(M)$.*

Here the index $\mathrm{ind}_x(v)$ is an integer (which we will not define here) which is non-zero only when v is zero at x. Thus for a surface S of genus g, this theorem says that $\sum_{x \in S} \mathrm{ind}_x(v) = 2 - 2g$. Thus if $g \neq 1$, then every vector field must have a point where it is zero, or else the sum on the right would be non-zero.

Corollary 3.7. *The only compact orientable surfaces that admit a Lie group structure are those with genus 1.*

Proof. If S forms a Lie group, then it is parallelizable and thus it has everywhere non-vanishing vector fields. By the Poincaré-Hopf theorem, this can only occur for tori. □

4. Claude Chevalley, Alexander Grothendieck and Michel Demazure

We will not discuss the work of Chevalley, Grothendieck, and Demazure at this time, except to say that they arguably brought Lie theory into its modern form through their work on algebraic groups. Specifically, Chevalley first redid almost the entire theory, then moved on with Borel to algebraic groups, which can be thought of as combining group and variety rather than group and manifold. Bringing in the techniques of algebraic geometry, he opened up a whole new chapter to Lie theory. Then Grothendieck and his student Demazure developed the entire theory over arbitrary base schemes in SGA 3 in the 1960s. We likely will not have much time to say much more about their highly interesting work (or about their highly interesting lives) in this class, but we may see a bit of it at the end.

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