~~g~~ $\mathfrak{sl}(2)$ and $\mathfrak{sl}(n)$

$\mathfrak{sl}(2)$ — traceless 2×2 $\quad$ $e = \begin{bmatrix}0&1\\0&0\end{bmatrix}$, $f = \begin{bmatrix}0&0\\1&0\end{bmatrix}$, $\overset{h}{\cancel{g}} = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$

Reps, $\quad ad(x)(y) = [x,y]$

important for $\mathfrak{sl}(n)$ as well.

Spaces

V_0		0		
V_1		-1	1	
V_2	2	0	-2	
V_3	3	1	-1	-3

Redefine Lie alg. reps.

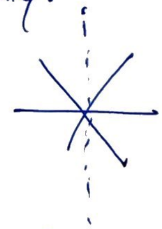
$Hw = \lambda_w w$

Fundamental construction: use $H(Ew) = (\lambda_w + 2)w$ and take highest weight — (doesn't exist for ∞-dim rep)

$\mathfrak{sl}(3)$: $\mathfrak{h} = \begin{bmatrix}1&&\\&-1&\\&&0\end{bmatrix}, \begin{bmatrix}1&&\\&0&\\&&-1\end{bmatrix}, \begin{bmatrix}0&&\\&1&\\&&-1\end{bmatrix}$ $\quad$ Cartan subalg.

ad rep:
$\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ $\qquad$ $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus \mathfrak{g}_\alpha$ $\quad$ root space where the α are functions on $\mathfrak{h}$ giving e-values.

The tricky thing is highest weight has 3 values potentially.
Resolve by setting hyperplane on the lattice



Steps to Construct ~~reps~~ $\mathfrak{sl}_3$ reps

Take $H \in \mathfrak{h}$, the subspace of diagonal matrices (dim 2)

$\mathfrak{sl}_2$ case | $[H,X] = 2X$, $[H,Y] = -2Y$, i.e, X and Y are eigenvectors for the adjoint action of H on ~~$\mathfrak{sp}_2\mathbb{C}$~~ $\mathfrak{sl}_2\mathbb{C}$

By "eigenvector" of $\mathfrak{h}$ now, we mean v s.t. $H(v) = \alpha(H)v \ \forall H \in \mathfrak{h}$.
with $\alpha \in \mathfrak{h}^*$. $\quad \mathfrak{sl}_3\mathbb{C} = \mathfrak{h} \oplus (\bigoplus \mathfrak{g}_\alpha)$

(This decomposition is a general result that any semisimple $\mathfrak{g}$, $\exists$ abelian $\mathfrak{h} \subset \mathfrak{g}$ s.t. $\mathfrak{h} \circlearrowright V$ (V a $\mathfrak{g}$-module) will be diagonalizable)

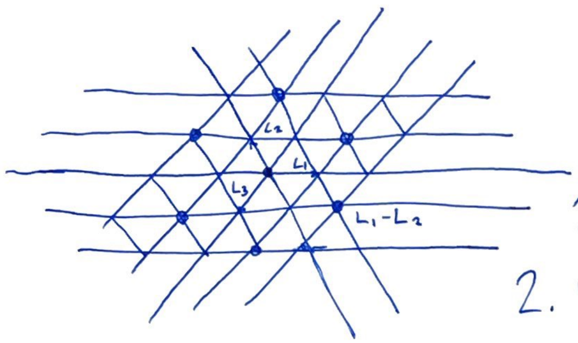
$[D,M] = (a_i - a_j)m_{ij} = km_{ij}$ iff $m_{ij} = E_{ij}$

So what is $[H,M]$? When is $[H,M] = \alpha M$?
H is diagonal, so we need for $M = E_{ij}$ $\quad$ ($E_{12} = \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}$ for instance)

Then the E_{ij} generate the eigenspaces for $\mathfrak{h} \overset{adj.}{\circlearrowright} \mathfrak{g}$.

Define functionals $L_i\begin{pmatrix}a_1&&\\&a_2&\\&&a_3\end{pmatrix} = a_i$ so $\alpha = L_i - L_j$.

Then clearly $\mathfrak{g}_{L_i - L_j}$ is generated by E_{ij}.



Here's the root lattice Λ_R

Two questions:

1. What is $\mathfrak{h} \curvearrowright \mathfrak{g}_\alpha$?
2. What is $\mathrm{ad}(X)Y$ for $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$?

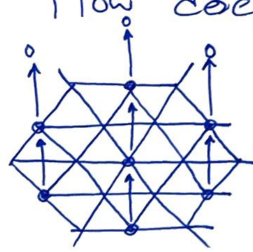
1. Is simple — $\mathfrak{h}$ acts triv. on $\mathfrak{g}_\alpha$, acting by α.
2. Requires a short calculation: How does H act on $\mathrm{ad}(X)Y$?

$$[H,[X,Y]] \overset{\text{Jacobi}}{=} [X,[H,Y]] + [[H,X],Y] = [X, \beta(H)\cdot Y] + [\alpha(H)\cdot X, Y]$$
$$= (\alpha(H) + \beta(H))\cdot [X,Y]$$

So $\mathrm{ad}(X)Y = [X,Y]$ is an eigenvector for $\mathfrak{h}$ with eigenvalue $\alpha + \beta$.

$$\mathrm{ad}(\mathfrak{g}_\alpha) : \mathfrak{g}_\beta \to \mathfrak{g}_{\alpha+\beta}$$

<u>Ex.</u> How does $\mathfrak{g}_{L_2 - L_3}$ act on Λ_R?



We're still left not knowing what $\ker(\mathrm{ad}(\mathfrak{g}_{L_2-L_3}))$ is.

Now what about <u>any</u> rep V of $\mathfrak{sl}_3$?

All we need is to know how H acts on $X(v)$ to know how X acts on v. ($H \in \mathfrak{h}$, $X \in \mathfrak{g}_\alpha$, $v \in V_\beta$).

$$HX(v) = XH(v) + [H,X](v) = X\beta(H)v + \alpha(H)Xv = (\alpha+\beta)Xv$$

so $X(v)$ is an eigenvector of $\mathfrak{h}$. So $\mathfrak{g}_\alpha$ sends V_β to $V_{\alpha+\beta}$ as expected.

Note: all α in an irrep. differ by lin. combinations of $L_i - L_j \in \mathfrak{h}^*$.

For sl_2, we used a highest weight vector to show the structure of any irrep. But what does 'highest' translate to here?

For this cause, we'll need some functional

$l : \Lambda \to \mathbb{R} \implies l : h^* \to \mathbb{C}$ and (we'll find out $\alpha \in \mathbb{R}$ always, in fact $\mathbb{Q}$ lin-span of Λ_R)

then choose the eigenspace V_α s.t. $\mathrm{Re}(l(\alpha))$ is maximal.

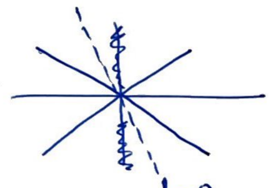
(we also need $\ker(l)$ empty, so l is irr. wrt. Λ_R)

Why is this at all useful?

Because now we can identify $v \in V_\alpha$ s.t.

v is an e-vector for h and $v \in \ker(g_\beta)$

<u>for $l(\beta) > 0$.</u>



Let's choose $l(a_1L_1 + a_2L_2 + a_3L_3) = pa_1 + qa_2 + ra_3$

with $p+q+r = 0$, $p > q > r$.

Q: What are g_α with $l(\alpha) > 0$?

ans: $g_{L_1-L_3}$, $g_{L_2-L_3}$, $g_{L_1-L_2}$.

(Try computing $l(\alpha)$ for $g_{L_1-L_3}$! $l(L_1-L_3) = p-r > 0$.)

We can now define the <u>highest weight vector</u> (again) it is:

$v \in V$, V finite-dim irrep, s.t. v is an e-vector of h and v is killed by E_{12}, E_{13}, E_{23}.

Nice. Now, where before we applied Y to such vectors killed by X, we now generate V irrep by applying E_{21}, E_{31}, E_{32}. (neg. root spaces)

<u>proof.</u> Want to show that $W = \{A \cdot v \mid A = E_{21}, E_{31}, E_{32}\}$ is invariant under sl_3 and so $W = V$.

Is $E_{21}v$ kept in W? $E_{12}E_{21}v = E_{21}E_{12}v + [E_{12}, E_{21}]v = \alpha([E_{12}, E_{21}])v$

and similarly $E_{23}E_{21}v = 0$. Induct on words with letters E_{21}, E_{32} of length n with corresponding W_n s.t. $W = \bigcup W_n$.

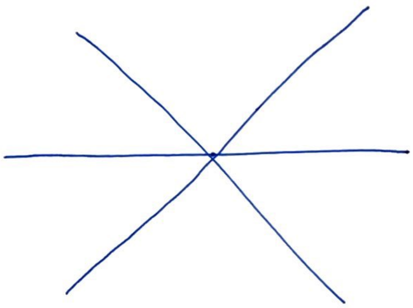
Then claim that E_{12}, E_{23} carry $W_n \to W_{n-1}$. $\sim \square$

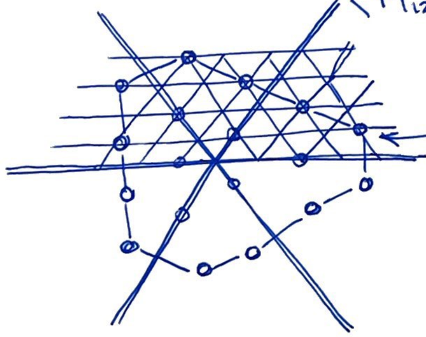
Details on p. 6

Prop. V any rep of $\mathfrak{sl}_3$, then for v highest weight,
$W = \{A \cdot v \mid A = \text{words in } E_{21}, E_{31}, E_{32}\}$ is irreducible.

Now let's consider the weight lattice.
~~Take some $\alpha = L_2 - L_1$ for example.~~

What is $E_{21}^k(v)$? These should lie in
$\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha + L_2 - L_1}, \ldots \mathfrak{g}_{\alpha + n(L_2 - L_1)}$ until it annihilates.





Where we can construct the rest of the diagram (S₃ sym?)
by swapping $p > q > r \to q > p > r$ and repeating.

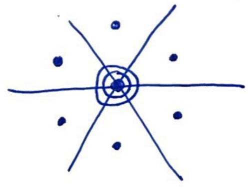
Prop. All e-values of any finite irrep. must lie in
$\Lambda_W \subset \mathfrak{h}^*$ generated by L_i and be cong. modulo
Λ_R generated by $L_i - L_j$.

Cor. $\Lambda_W / \Lambda_R \cong \mathbb{Z}/3\mathbb{Z}$.

Here's the nice conclusion: $E_{12}, E_{21}, [E_{12}, E_{21}]$ for instance span
a subalgebra iso. to $\mathfrak{sl}_2$.

Ex. Take $V = \mathbb{C}^3$ standard rep. weight diagram: 

Or now $V \otimes V^*$, rep'd by sums of weights L_i
from V and $-L_j$ of V^*.

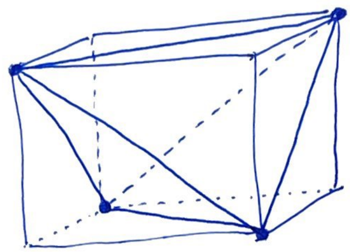


This is not an irrep since we
have $v \otimes u^* \mapsto u^*(v)$, the adj rep of
$\mathfrak{sl}_3$ which is irr. Important in physics!

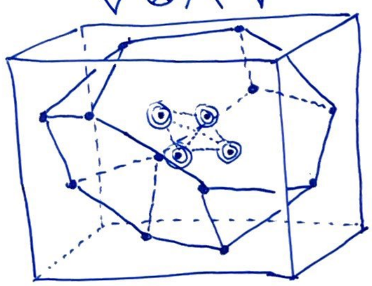
$\mathfrak{sl}_4$ and $\mathfrak{sl}_n$ at last!

standard ; V^* is simply $-L_i$

$\mathfrak{sl}_4$ diagrams :



$V \otimes \Lambda^2 V$



Not irr.

$L_i + (L_i + L_j)$, $L_i + L_j + L_k$

$2L_i + L_j$

occur once

occur 3 times

$\mathfrak{sl}_n$ is a quick generalization.

V standard rep has highest weight L_1,

$\Lambda^k V$ is irr. with highest weight $L_1 + \cdots + L_k$.

Then irrep. $\Gamma_{a_1, \ldots a_{n-1}}$ with highest weight

$(a_1 + \cdots + a_{n-1}) L_1 + \cdots + a_{n-1} L_{n-1} = \sum_{i=1}^{n-1} \left(\sum_{j=i}^{n-1} a_j \right) L_i$ will

appear in $\bigotimes_{i=1}^{n-1} \mathrm{Sym}^{a_i}(\Lambda^i V)$

Again E_{ij} ~~for a~~ is in $\mathrm{End}(\mathbb{C}^n)$ and sends

$e_j \mapsto e_i$ and kills e_k for $k \neq j$.

So E_{ij} is an e-vector for $\mathfrak{h}$ with e-value $L_i - L_j$.

The roots are thus pairwise differences $L_i - L_j$.

$\Lambda_W / \Lambda_R \cong \mathbb{Z}/n\mathbb{Z}$

Write such a word as $w_n = E_{21} \circ w_{n-1}$ or $E_{32} \circ w_{n-1}$.

$w_{n-1}(v)$ is an e-vector with e-value β. For $E_{21} \circ w_{n-1}$,

$$E_{12} w_n(v) = E_{12} E_{21} w_{n-1}(v) \in E_{21}(W_{n-2}) + \beta([E_{12}, E_{21}])\, w_{n-1}(v)$$

$$\subset W_{n-1}$$

and $$E_{23}(E_{21} w_{n-1}(v)) = E_{21} E_{23} w_{n-1}(v) + [E_{23}, E_{21}]\, w_{n-1}(v)$$

$$\in E_{21} W_{n-2} \subset W_{n-1}$$

and same for $E_{32} \circ w_{n-1}$.