Roots and the Weyl Group

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Contents

1 The Conjugation Action and Its Differential

Definition 1.1 (Conjugation Map)**.** Let G be a Lie group. For each $g \in G$, define the *conjugation map*

$$c_g : G \to G, \quad c_g(h) = ghg^{-1}.$$

Definition 1.2 (Adjoint Representation)**.** Let G be a Lie group with Lie algebra $\mathfrak{g}$. The *adjoint representation* of G is the map

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}), \quad \mathrm{Ad}(g) = d(c_g)_e,$$

where $d(c_g)_e$ is the differential of c_g at the identity e of G.

Proposition 1.3. $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ *is a Lie group homomorphism; that is, for all* $g_1, g_2 \in G$*,*

$$\mathrm{Ad}(g_1 g_2) = \mathrm{Ad}(g_1)\,\mathrm{Ad}(g_2).$$

Proof. For any $g_1, g_2 \in G$ and $h \in G$, we have:

$$c_{g_1 g_2}(h) = g_1 g_2 h g_2^{-1} g_1^{-1} = c_{g_1}\big(c_{g_2}(h)\big).$$

Taking the differential at e and applying the chain rule yields:

$$d(c_{g_1 g_2})_e = d(c_{g_1})_e \circ d(c_{g_2})_e,$$

so that

$$\mathrm{Ad}(g_1 g_2) = \mathrm{Ad}(g_1)\,\mathrm{Ad}(g_2).$$

□

2 Linear Actions of S^1 and Tori

2.1 Complex Actions of S^1

Lemma 2.1. *Let* S^1 *act linearly on a finite-dimensional complex vector space* V*. Then there exists a basis in which the action is diagonal. Specifically, for* $z = e^{i\theta} \in S^1$*,*

$$z \cdot v = \mathrm{diag}(e^{in_1\theta}, e^{in_2\theta}, \ldots, e^{in_k\theta})\, v,$$

for some integers $n_1, \ldots, n_k$*.*

Definition 2.2. A *character* of S^1 is a group homomorphism $\chi : S^1 \to S^1$. Every character is of the form

$$\chi(e^{i\theta}) = e^{in\theta}$$

for some $n \in \mathbb{Z}$.

Remark 2.3. Since every complex representation of S^1 is diagonalizable, it follows that any finite-dimensional complex representation of a maximal torus T (which is isomorphic to a product of copies of S^1) decomposes as a direct sum of one-dimensional subrepresentations (i.e., characters).

3 Weight Decomposition and Roots

3.1 Restriction to a Maximal Torus

Let T be a maximal torus in G with Lie algebra $\mathfrak{t}$. Since T is abelian, every finite-dimensional complex representation of T decomposes into one-dimensional subrepresentations (characters). In particular, the restriction of the adjoint representation of G to T yields a decomposition of $\mathfrak{g}$ into eigenspaces.

Proposition 3.1 (Weight Decomposition)**.** *Let G be a compact Lie group and T a maximal torus in G. Then the Lie algebra $\mathfrak{g}$ decomposes as*

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\lambda \in \Lambda \setminus \{0\}} \mathfrak{g}_\lambda,$$

where for each linear functional $\lambda \in \mathfrak{t}^*$,

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : \mathrm{Ad}(t)X = e^{2\pi i\, \lambda(\log t)} X \quad \textit{for all } t \in T\},$$

and Λ is the set of all weights.

3.2 Definition of Roots

Definition 3.2 (Roots)**.** A nonzero weight $\alpha \in \mathfrak{t}^*$ is called a *root* of G (or of $\mathfrak{g}$) with respect to T if

$$\mathfrak{g}_\alpha \neq \{0\}.$$

The set of all roots is denoted by $\Phi \subset \mathfrak{t}^* \setminus \{0\}$.

4 The Weyl Group

4.1 Definition

Definition 4.1 (Normalizer and Weyl Group)**.** Let T be a maximal torus in a connected, compact Lie group G. The *normalizer* of T is defined by

$$N_G(T) = \{g \in G : gTg^{-1} = T\}.$$

Since T is abelian, it is a normal subgroup of $N_G(T)$. The *Weyl group* of G with respect to T is the quotient group:

$$W(G,T) = N_G(T)/T.$$

Proposition 4.2. *The Weyl group $W(G,T)$ is finite.*

Proof. Define the natural surjection

$$\frac{N(T)}{T} \twoheadrightarrow \frac{N(T)}{N_0(T)}, \quad \ker \;=\; \frac{N_0(T)}{T}.$$

To show $N_0(T) = T$, suppose, for contradiction, that $N_0(T) \supsetneq T$.
Then there exists

$$X \;\in\; \mathrm{Lie}\big(N_0(T)\big) \setminus \mathfrak{t}.$$

Since T is abelian and $N_0(T)$ is connected, the conjugation action of $N_0(T)$ on $\mathfrak{t}$ is trivial, implying

$$[X, \mathfrak{t}] \;=\; 0.$$

Hence $\mathfrak{t} + \mathbb{R}X$ is an abelian Lie subalgebra. Set

$$A = \exp\big(\mathfrak{t} + \mathbb{R}X\big),$$

which is a connected abelian subgroup strictly containing T, contradicting the maximality of T.

Thus $N_0(T) = T$.

Consequently,

$$W(G,T) = \frac{N(T)}{T} = \frac{N(T)}{N_0(T)},$$

a finite group. □

4.2 Examples

4.2.1 Type A: $G = U(n)$

Example 4.3. Let $G = U(n)$ be the group of $n \times n$ unitary matrices.

A standard maximal torus is

$$T = \{\mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n}) : \theta_j \in \mathbb{R}\}.$$

Its Lie algebra is

$$\mathfrak{t} = \{\mathrm{diag}(i\theta_1, i\theta_2, \dots, i\theta_n) : \theta_j \in \mathbb{R}\}.$$

The adjoint representation of $U(n)$ restricted to T yields the weight spaces with nonzero weights given by:

$$\lambda_{jk}(H) = \theta_j - \theta_k, \quad 1 \le j \ne k \le n.$$

These nonzero weights form the root system of $U(n)$.

Moreover, the normalizer $N_{U(n)}(T)$ consists of all monomial unitary matrices, so that

$$W(U(n), T) \cong S_n,$$

the symmetric group on n letters.

4.2.2 Type B: $G = SO(2n+1)$

Example 4.4. Let $G = SO(2n+1)$ be the group of $(2n+1) \times (2n+1)$ real orthogonal matrices with determinant 1.

A maximal torus T can be chosen as

$$T = \left\{ \begin{pmatrix} R(\theta_1) & & & 0 \\ & R(\theta_2) & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} : \theta_j \in \mathbb{R} \right\},$$

where each $R(\theta_j)$ is a 2×2 rotation matrix:

$$R(\theta_j) = \begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix}.$$

The Weyl group in this case is

$$W(SO(2n+1), T) \cong (\mathbb{Z}_2)^n \rtimes S_n,$$

where the factor $(\mathbb{Z}_2)^n$ corresponds to sign changes (reflections) in each rotation block and S_n permutes the blocks.

4.2.3 Type C: $G = Sp(n)$

Example 4.5. Let $G = Sp(n)$ be the compact symplectic group, defined by

$$Sp(n) = \{A \in U(2n) : A^T J A = J\},$$

where

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

A maximal torus in $Sp(n)$ is given by

$$T = \left\{\operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}, e^{-i\theta_1}, \ldots, e^{-i\theta_n}) : \theta_j \in \mathbb{R}\right\}.$$

Its Lie algebra is

$$\mathfrak{t} = \{\operatorname{diag}(i\theta_1, \ldots, i\theta_n, -i\theta_1, \ldots, -i\theta_n) : \theta_j \in \mathbb{R}\}.$$

The weight decomposition of the adjoint representation shows that the roots are of the form:

$$\pm 2\epsilon_i \quad \text{and} \quad \pm(\epsilon_i \pm \epsilon_j) \quad (1 \leq i < j \leq n).$$

The Weyl group is again isomorphic to

$$W(Sp(n), T) \cong (\mathbb{Z}_2)^n \rtimes S_n.$$

4.2.4 Type D: $G = SO(2n)$

Example 4.6. Let $G = SO(2n)$ be the group of $2n \times 2n$ real orthogonal matrices with determinant 1.

A maximal torus T in $SO(2n)$ is given by block-diagonal matrices with 2×2 rotation blocks:

$$T = \{\operatorname{diag}(R(\theta_1), R(\theta_2), \ldots, R(\theta_n)) : \theta_j \in \mathbb{R}\}.$$

Its Lie algebra is

$$\mathfrak{t} = \{\operatorname{diag}(\begin{pmatrix} 0 & -\theta_1 \\ \theta_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -\theta_n \\ \theta_n & 0 \end{pmatrix})\}.$$

The weight decomposition of the adjoint representation yields roots of the form

$$\pm(\epsilon_i \pm \epsilon_j), \quad 1 \leq i < j \leq n.$$

The Weyl group in this case is isomorphic to

$$W(SO(2n), T) \cong (\mathbb{Z}_2)^{n-1} \rtimes S_n,$$

reflecting the fact that among the n sign changes, only an even number are permitted (to preserve the determinant 1 condition).

4.2.5 Type E: $G = E_6$

Example 4.7. The exceptional Lie group E_6 is a compact simply connected Lie group with a maximal torus of dimension 6 and the Weyl group of E_6 is defined as

$$W(E_6, T) = N_{E_6}(T)/T,$$

and it is a finite group.

This Weyl group acts on the Cartan subalgebra (of dimension 6) by reflections and encodes the symmetry of the root system of E_6.

References

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[2] J. Morgan, *Lie Groups Lecture 7: Tori and Weyl Groups* (2024), https://www.math.columbia.edu/~jmorgan/2024LieGroups/2024LGLecture7.pdf