Weyl Character Formula and BGG Resolution

Yan

April 1, 2025

Contents

1 Dominant Integral Weights and BGG Resolution

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$, and let R^+ denote the set of positive roots. Recall the set of dominant integral weights:

$$P_+ = \{\lambda \in \mathfrak{h}^* \mid \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}_{\geq 0},\ \forall \alpha \in R^+\}.$$

For each $\lambda \in P_+$, the BGG resolution constructs an exact sequence involving Verma modules M_μ and irreducible representations L_λ:

$$0 \to M_{w_0 \cdot \lambda} \to \cdots \to \bigoplus_{\ell(w)=k} M_{w \cdot \lambda} \to \cdots \to \bigoplus_{\ell(w)=1} M_{w \cdot \lambda} \to M_\lambda \to L_\lambda \to 0,$$

where:

- w_0 is the longest element in the Weyl group W.
- $\ell(w)$ is the minimal number of simple reflections required to express w.
- The affine action $w \cdot \lambda$ is defined by:

$$w \cdot \lambda = w(\lambda + \rho) - \rho, \quad \rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha.$$

Example 1.1 (BGG Resolution for $\mathfrak{sl}(2)$)**.** Let $\mathfrak{g} = \mathfrak{sl}(2)$, and let $\lambda = \frac{n}{2}\alpha$ for $n \in \mathbb{Z}_{\geq 0}$. Then the BGG resolution becomes:

$$0 \to M_{-\frac{n+2}{2}\alpha} \to M_{\frac{n}{2}\alpha} \to L_{\frac{n}{2}\alpha} \to 0.$$

2 Characters and Weyl Character Formula

Definition 2.1 (Character)**.** Given a $\mathfrak{g}$-module V, its character is defined by:

$$\mathrm{ch}(V) = \sum_{\lambda \in \mathfrak{h}^*} \dim(V[\lambda]) e^{\lambda},$$

where $V[\lambda]$ is the weight space corresponding to λ.

Theorem 2.2 (Theorem 8.33)**.** *For $\lambda \in P_+$, we have:*

$$\mathrm{ch}(L_\lambda) = \sum_{w \in W} (-1)^{\ell(w)} \mathrm{ch}(M_{w \cdot \lambda}).$$

Proof. Apply the character functor $\mathrm{ch}(\cdot)$ to the BGG resolution. Characters are additive, hence we obtain an alternating sum of Verma module characters. □

Theorem 2.3 (Weyl Character Formula, Theorem 8.34)**.** *Explicitly, the character of an irreducible representation L_λ is:*

$$\mathrm{ch}(L_\lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)}}{\prod_{\alpha \in R^+} (e^{\alpha/2} - e^{-\alpha/2})}.$$

Proof. Substitute the character of a Verma module:

$$\mathrm{ch}(M_\mu) = \frac{e^{\mu}}{\prod_{\alpha \in R^+} (1 - e^{-\alpha})},$$

into Theorem 8.33 to directly derive the Weyl character formula. □

Remark 2.4 (Weyl Denominator Identity)**.** The denominator is simplified by the Weyl denominator identity:

$$\prod_{\alpha \in R^+} (e^{\alpha/2} - e^{-\alpha/2}) = \sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)}.$$