Math UN1101: Calculus I, Section 1
Midterm 1 Solutions

Instructions

- Write **your name and UNI** clearly in the section below.
- You are **NOT** allowed to use class notes, books and homework solutions in the examination.
- Except for True/False questions, show all computations and work in your answer.
- **Don't cheat**! If it looks like you are cheating, then you are cheating.

Question	Points	Score
1	10	
2	10	
3	4	
4	6	
5	10	
6	5	
7	5	
Total:	50	

Name:______________________________

UNI:______________________________

1. (10 points) **True/False** 2 points each

   (a) T F $f(x) = \sin^3(x)$ is an even function.

   (b) T F The graph of $f(-x)$ is obtained from reflecting the graph of $f(x)$ about the $y-$axis.

   (c) T F We have that

$$\lim_{x\to 0} \frac{\sin\left(\frac{1}{x}\right)}{x+1} = \frac{\lim_{x\to 0} \sin\left(\frac{1}{x}\right)}{\lim_{x\to 0} x+1}$$

   (d) T F The function $f(x) = x^3 - 3x - 1$ has a root in $(-1, 0)$.

   (e) T F The derivative of 2^2 is $2 \cdot 2 = 4$.

(You may use this area as scratchwork.)

Solution:

(a) **F**. We compute that $f(-x) = \sin^3(-x) = (-\sin(x))^3 = -\sin^3(x) = -f(x)$. Therefore $f(x) = \sin(x)$ is odd, not even.

(b) **T**. The graph of $f(-x)$ is obtained from reflecting the graph of $f(x)$ about the $y-$axis.

(c) **F**. Because

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) \text{ DOES NOT EXIST}$$

we cannot use the Quotient limit law and thus the equation above is false.

(d) **T**. We compute that $f(-1) = 1$ and $f(0) = -1$. Because polynomials are continuous at all real numbers and in particular in the interval $[-1, 0]$ the Intermediate Value Theorem shows that $f(x)$ must equal 0 at some point in $(-1, 0)$ and therefore $f(x)$ has a solution in $(1, 0)$.

(e) **F** $2^2 = 4$ is a constant. Therefore the derivative of 2^2 is zero. You cannot apply the Power Rule here.

2. Compute the following limits, if they exist. If the limit does not exist, explain why.

(a) (3 points) $\lim_{x\to 2} \frac{x-2}{x^2-x-2}$

Solution:

$$\lim_{x\to 2} \frac{x-2}{x^2-x+-2} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+1)} \overset{x\neq 2}{=\!=} \lim_{x\to 2} \frac{1}{(x+1)} = \frac{1}{3}$$

where we have used the fact that because $\frac{1}{(x+1)}$ is continuous at $x=2$, we can substitute $x=2$ in to evaluate the limit.

(b) (3 points) $\lim_{x\to 0} \sin^2(x)\cos\left(\frac{1}{x}\right)$

Solution: Notice that $-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$. Because $\sin^2(x) \geq 0$ for any value of x, it follows that we have the inequality

$$-\sin^2(x) \leq \sin\left(\frac{1}{x}\right) \leq \sin^2(x)$$

Notice that $\lim_{x\to 0} -\sin^2(x) = \lim_{x\to 0} \sin^2(x) = \pm\sin^2(0) = 0$ because $\sin(x)$ is continuous so we can just plug in 0 to evaluate the limit. Therefore by the Squeeze Theorem it follows that

$$\lim_{x\to 0} x^4 \sin\left(\frac{1}{x}\right) = 0$$

(c) (4 points) $\lim_{x\to 0} 4^{\frac{\sqrt{4+x}-\sqrt{4-x}}{x}}$

Solution: Because 4^x is continuous at all real numbers, and continuous functions commute with limits, we can bring the limit inside, e.g.

$$\lim_{x\to 0} 4^{\frac{\sqrt{4+x}-\sqrt{4-x}}{x}} = 4^{\lim_{x\to 0}\frac{\sqrt{4+x}-\sqrt{4-x}}{x}} \tag{1}$$

We now compute the limit inside by rationalizing the numerator.

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{4+x}-\sqrt{4-x}}{x} &= \lim_{x\to 0}\frac{\sqrt{4+x}-\sqrt{4-x}}{x}\cdot\frac{\sqrt{4+x}+\sqrt{4-x}}{\sqrt{4+x}+\sqrt{4-x}}\\ &= \lim_{x\to 0}\frac{(\sqrt{4+x})^2-(\sqrt{4-x})^2}{x(\sqrt{4+x}+\sqrt{4-x})} = \lim_{x\to 0}\frac{(4+x)-(4-x)}{x(\sqrt{4+x}+\sqrt{4-x})}\\ &= \lim_{x\to 0}\frac{2x}{x(\sqrt{4+x}+\sqrt{4-x})} \overset{x\neq 0}{=\!=} \lim_{x\to 0}\frac{2}{(\sqrt{4+x}+\sqrt{4-x})}\end{aligned}$$

Now notice that function in the final expression above is continuous at $x=0$ because the denominator is not 0. Therefore by continuity we can plug in 0 to evaluate the limit and find that

$$\lim_{x\to 0}\frac{\sqrt{4+x}-\sqrt{4-x}}{x} = \frac{2}{\sqrt{4}+\sqrt{4}} = \frac{2}{4} = \frac{1}{2}$$

To obtain the final answer we plug this back into Equation (1) and find

$$\lim_{x\to 0} 4^{\frac{\sqrt{4+x}-\sqrt{4-x}}{x}} = 4^{\frac{1}{2}} = 2$$

3. Please give formal definitions below.

(a) (2 points) What does it mean for a function $f(x)$ to be continuous at a point a?

Solution: $f(x)$ is continuous at a point a if the both conditions are satisfied

- $\lim_{x\to a} f(x)$ exists
- $\lim_{x\to a} f(x) = f(a)$ ($f(x)$ has the Direct Substitution Property at a.)

(b) (2 points) What does it mean for a function $f(x)$ to be differentiable at a point a?

Solution: $f(x)$ is differentiable at the point a if

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \text{ exists}$$

Specifically, the limit above exists if and only if the left-handed limit equals the right-handed limits. This means that $f(x)$ is differentiable at the point a if

$$\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a} = \lim_{\substack{h\to 0\\ h>0}}\frac{f(a-h)-f(a)}{-h} = \lim_{\substack{h\to 0\\ h>0}}\frac{f(a+h)-f(a)}{h} = \lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}$$

4. Consider the following function.

$$f(x) = \begin{cases} 2 & \text{if } x \leq -1 \\ x^2 + 1 & \text{if } -1 < x < 3 \\ \dfrac{1}{2-x} & \text{if } x \geq 3 \end{cases}$$

(a) (3 points) For what values of x is f not continuous at x?

Solution: $f(x)$ is not continuous at $x = 3$. To check for continuity, look at the criteria from $3(a)$.

2, $x^2 + 1$ and $\dfrac{1}{2-x}$ are all continuous in the regions prescribed above, so we just need to check if $f(x)$ is continuous where they meet, aka at $x = -1, 3$. We first check $x = -1$

$$\lim_{x \to -1^-} 2 \overset{?}{=\!=} \lim_{x \to -1^+} x^2 + 1$$

Individually on their own, both functions are continuous at $x = -1$, so to evaluate the one-sided limits is the same as evaluating the limit by plugging in -1. Thus we see that $2 = (-1)^2 + 1 = 1 + 1 = 2$ and so $\lim_{x \to -1} f(x)$ exists -1. Moreover the Direct Substitution Property is automatic, as $f(-1) = 2 = \lim_{x \to -1^-} 2$. Thus $f(x)$ is continuous at $x = -1$.

We repeat the same calculation for $x = 3$. Again since x^2+1 and $\dfrac{1}{4-x}$ are continuous at $x = 3$ we can just plug in 3 to evaluate the one-handed limits.

$$10 = 3^2 + 1 = \lim_{x \to 3^-} x^2 + 1 \neq \lim_{x \to 3^+} \frac{1}{2-x} = \frac{1}{2-3} = -1$$

Thus we see that $\lim_{x \to 3} f(x)$ doesn't exist so it already fails the first criteria listed in $3(a)$. Thus $f(x)$ is not continuous at $x = 3$.

Many people put down that $f(x)$ is not continuous at 2 also. But notice at 2, $f(x)$ is of the form $x^2 + 1$ and that $f(x) = \dfrac{1}{2-x}$ only when $x \geq 3$.

(b) (3 points) For what values of x is f not differentiable at x?

Solution: $f(x)$ is not differentiable at $x = -1, 3.$

Here one can use the result that if a function $f(x)$ is differentiable at a, then it must be continuous at a. Notice this means that if $f(x)$ is not continuous at a, it is not differentiable at a. Thus right from the start we know that $f(x)$ is not differentiable at $x = 3$. Like before outside these values and at $x = -1$ $f(x)$ is either a constant, a polynomial or a rational function and so is differentiable. It remains to check $x = -1$. Using $3(b)$ we need to check if

$$\lim_{\substack{h \to 0 \\ h > 0}} \frac{f(-1-h) - f(-1)}{-h} \overset{?}{=\!=} \lim_{\substack{h \to 0 \\ h > 0}} \frac{f(-1+h) - f(-1)}{h}$$

Recall that $f(-1) = 2$. Because $-1 - h < -1$ for $h > 0$, by definition, $f(x) = 2$ so the left handed side above is then

$$\lim_{\substack{h \to 0 \\ h > 0}} \frac{f(-1-h) - f(-1)}{-h} = \lim_{\substack{h \to 0 \\ h > 0}} \frac{2-2}{-h} = \lim_{\substack{h \to 0 \\ h > 0}} \frac{0}{-h} = 0$$

We repeat the same for the right hand side above where now $-1 + h > 3$ for $h > 0$ and so $f(x) = x^2 + 1$ and find

$$\lim_{\substack{h \to 0 \\ h > 0}} \frac{f(-1+h) - f(-1)}{h} = \lim_{\substack{h \to 0 \\ h > 0}} \frac{((-1+h)^2 + 1) - 2}{h} = \lim_{\substack{h \to 0 \\ h > 0}} \frac{((1 - 2h + h^2) + 1) - 2}{h}$$

$$= \lim_{\substack{h \to 0 \\ h > 0}} \frac{-2h + h^2}{h} = \lim_{\substack{h \to 0 \\ h > 0}} \frac{h(-2+h)}{h} \overset{h \neq 0}{=\!=} \lim_{\substack{h \to 0 \\ h > 0}} -2 + h \overset{cont}{=\!=} -2$$

Since the left and right handed limits (of the slopes) don't agree at $x = -1$ we see that $f(x)$ is not differentiable at $x = -1$ as well.

5. Compute the value of the derivative of $f(x)$ at the point a. If $f(x)$ is not differentiable at a, explain why.

(a) (3 points) $f(x) = x^2 - 3\sqrt{x}$, $a = 9$

Solution: Write $f(x) = x^2 - 3x^{\frac{1}{2}}$ and using the power rule we see that

$$f'(x) = 2x - 3\frac{1}{2}x^{(\frac{1}{2}-1)} = 2x - \frac{3}{2}x^{-\frac{1}{2}} = 2x - \frac{3}{2\sqrt{x}}$$

Plugging in $a = 9$ we see that

$$f'(9) = 2(9) - \frac{3}{2\sqrt{9}} = 18 - \frac{1}{2} = 17.5$$

(b) (3 points) $f(x) = \dfrac{-4}{x^5}$, $a = 1$

Solution: Write $f(x) = -4x^{-5}$ and using the power rule we see that

$$f'(x) = -4(-4)x^{(-5-1)} = 20x^{-6} = \frac{20}{x^6}$$

Plugging in $a = 1$ we see that

$$f'(1) = \frac{20}{1^6} = 20$$

(c) (4 points) $f(x) = x|x|$, $a = 0$

Solution: We can't use the power rule here since $|x| \neq x$. Thus we need to use the definition of the derivative. In fact we claim that $f(x)$ is differentiable at $a = 0$ and that $f'(0) = 0$. Using $3(b)$ We compute the right and left-handed limits of

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$$

As $|-h| = h$ for $h > 0$, we see that

$$L \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{\substack{h\to 0\\ h>0}} \frac{f(0-h) - f(0)}{-h} = \lim_{\substack{h\to 0\\ h>0}} \frac{(-h)|-h| - 0}{-h}$$

$$= \lim_{\substack{h\to 0\\ h>0}} \frac{-h(h)}{-h} \stackrel{h\neq 0}{=\!=} \lim_{\substack{h\to 0\\ h>0}} h = 0$$

As $|h| = h$ for $h > 0$ we see that

$$R \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{\substack{h\to 0\\ h>0}} \frac{f(0+h) - f(0)}{h} = \lim_{\substack{h\to 0\\ h>0}} \frac{(h)|h| - 0}{h}$$

$$= \lim_{\substack{h\to 0\\ h>0}} \frac{h(h)}{h} \stackrel{h\neq 0}{=\!=} \lim_{\substack{h\to 0\\ h>0}} h = 0$$

Since the left and right handed limits agree we see that $f(x)$ is differentiable at $a = 0$ and $f'(0) = 0$!

6. (5 points) Find an equation of the tangent line to the curve $y = 2x^3 - 19x + 20$ at the point $(2, -2)$.

Solution: By definition, the equation of the tangent line at $(a, f(a))$ is the line

$$y - f(a) = f'(a)(x - a)$$

We compute that $f'(x) = 6x^2 - 19$ by the power rule and thus $f'(2) = 24 - 19 = 5$. Therefore the equation of the tangent line is

$$y - (-2) = 5(x - 2) \implies y = 5x - 12$$

7. (5 points) Find all vertical and horizontal asymptotes of the graph of $f(x) = \dfrac{\sqrt{4x^2 - 1} + x}{4x - 1}$.

Solution: We first compute the horizontal asymptotes. Recall that $\sqrt{x^2} = |x|$. Thus as x goes to positive ∞ we have that $\sqrt{x^2} = x$ and therefore

$$\lim_{x\to\infty} \frac{\sqrt{4x^2-1}+x}{4x-1} = \lim_{x\to\infty} \frac{\sqrt{4x^2-1}/x + x/x}{4 - 1/x} = \lim_{x\to\infty} \frac{\sqrt{4x^2-1}/\sqrt{x^2} + x/x}{4 - 1/x}$$
$$= \frac{\lim_{x\to\infty} \sqrt{4 + {-1}/x^2} + 1}{\lim_{x\to\infty} 4 - 1/x} = \frac{\sqrt{4-0}+1}{4-0} = \frac{3}{4}$$

Now as x goes to negative ∞, we have that $\sqrt{x^2} = |x| = -x$ as x is negative. Thus it follows that $x = -\sqrt{x^2}$ in this case and we find that

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2-1}+x}{4x-1} = \lim_{x\to-\infty} \frac{\sqrt{4x^2-1}/x + x/x}{4 - 1/x} = \lim_{x\to-\infty} \frac{\sqrt{4x^2-1}/-\sqrt{x^2} + x/x}{4 - 1/x}$$
$$= \frac{\lim_{x\to-\infty} -\sqrt{4 + {-1}/x^2} + 1}{\lim_{x\to-\infty} 4 - 1/x} = \frac{-\sqrt{4-0}+1}{4-0} = \frac{-1}{4}$$

Be careful! Notice that only the term under the square root in the numerator changes sign. Thus the horizontal asympototes are at $\boxed{y = \frac{3}{4} \text{ and at } y = \frac{-1}{4}}$.

The vertical asympototes are where the denominator of $f(x)$ is zero. This happens exactly when $4x - 1 = 0 \implies \boxed{x = \frac{1}{4}}$ is the vertical asymptote.