Bruhat Order Let $u, w \in W$. Then

(I) $u \xrightarrow{t} w$ implies that $ut = w$ for $t \in T$ and $\ell(u) < \ell(w)$

(II) $u \to w$ implies that $u \xrightarrow{t} w$ for some $t \in T$

(III) $u \le w$ means that there exist $u_i \in W$ s.t.

$$u = u_0 \to u_1 \longrightarrow \dots \longrightarrow u_k = w$$

The subword property and the chain property are consequences of the following lemma.

lemma

For $u, w \in W$, $u \neq w$, let $w = s_1 s_2 \dots s_q$ be reduced, and suppose that some reduced expression for u, is a subword of $s_1 s_2 \dots s_q$. Then $\exists v \in W$ s.t.

(I) $v > u$

(II) $\ell(v) = \ell(u) + 1$

(III) some reduced expression for v is a subword of $s_1 s_2 \dots s_q$

proof Of all reduced subword expressions $u = s_1 \dots \widehat{s_{i_1}} \dots \widehat{s_{i_k}} \dots s_q$, $1 \le i_1 < \dots < i_k \le q$. choose one st i_k is minimal. Let

$$t = s_q s_{q-1} \dots s_{i_k} \dots s_{q-1} s_q$$

then $ut = s_1 \dots \widehat{s_{i_1}} \dots \widehat{s_{i_{k-1}}} \dots s_{i_k} \dots s_q$ so $\ell(ut) \le \ell(u) + 1$. We claim that $ut > u$. If so, $v = ut$ satisfies (I) - (III) $\square$

In addition to this lemma we also need the following properties

(1) exchange If $w = s_1 \dots s_k$ and $\ell(wt) < \ell(w)$ $(w \in W, t \in T)$ then

- $wt = s_1 \dots \hat{s}_i \dots s_k$ (for some i)
- $t = s_k \dots s_i \dots s_k$

(2) deletion If $w = s_1 \dots s_k$ and $\ell(w) < k$ then $w = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_k$ (for some i, j).

Subword Property

Let $v = s_1 \dots s_q$ be a reduced subword for V. then

$$u \le v \leftrightarrow u = \overbrace{s_{i_1} s_{i_2} \dots s_{i_k}}^{\text{subword of } s_1 \dots s_q}$$

for some $1 \le i_1 < \dots < i_k \le q$

proof

$(\Rightarrow)$ Assume $u \le w$, then we have:

$$u = u_0 \xrightarrow{t} u_1 \dots \xrightarrow{t} u_m = w$$

then $u_{m-1} = wt_m = s_1 \dots \hat{s}_i \dots s_q$ for some i due to the exchange property stated above. If you repeat this argument to $u_{m-2} \dots u_0$, then we have an expression of u that is a subword of w. By the deletion property, we know that as a subword, it contains a reduced expression of u.

($\Leftarrow$) If u has a reduced exp. that is a subword of $s_1 s_2 \dots s_q$ then the above lemma allows us to construct the sequence $u < v_1 < \dots < v_s$ st. thier lengths are strictly increasing by one, but each has a reduced word that is a subword of $s_1 s_2 \dots s_q$. Then it is clear that $v_s = w$. $\square$

Corollary for $u, w \in W$, the following are equivalent

(i) $u \le w$

(ii) every reduced expression for w has a subword that is a reduced expression for u.

(iii) some reduced expressions for w has a subword that is a reduced expression for u. $\square$

Corollary Bruhat intervals $[u, w]$ are finite. card $[u, w] \le 2^{\ell(w)}$.

Corollary The mapping $w \mapsto w^{-1}$ is an automorphism of Bruhat order (ie. $u \le w \Leftrightarrow u^{-1} \le w^{-1}$).

Chain Property If $u < w$, there exists a chain $u = x_0 < x_1 < \dots < x_k = w$ s.t. $\ell(x_i) = \ell(u) + i$, for $1 \le i \le k$.

proof this follows directly from the first lemma and the subword property $\square$

Lifting Property Suppose $u < w$ and $s \in D_L(w) \setminus D_L(u)$. Then $u \le sw$ and $su \le w$

proof let $\alpha \prec \beta$ denote the subword relation between a word β and a subword α. Choose a reduced decomposition $sw = s_1 s_2 \dots s_q$. Then, $w = s s_1 s_2 \dots s_q$ is also reduced, and there exists a reduced subword

$$u = s_{i_1} s_{i_2} \dots s_{i_k} \prec s s_1 s_2 \dots s_q$$

Now, $s_{i_1} \neq s$ since $su > u$; hence

$$s_{i_1} s_{i_2} \dots s_{i_k} \prec s_1 s_2 \dots s_q \Rightarrow u \le sw$$

and

$$s s_{i_1} s_{i_2} \dots s_{i_k} \prec s s_1 s_2 \dots s_q \Rightarrow su \le w \quad \square$$