(1)

Parabolic Subgroups and the Tableau Criterion

Ex 1. $n=3$ Is $213 < 312$ in Bruhat order? $x=213$, $y=312$

note: For S_n $\pi = abc\ldots$, $D_R(\pi) = \{i \mid \pi(i) > \pi(i+1)\}$

1. Find $D_R(x) = \{1\}$ (index where x decreases)
2. Take the first i digits of x and y, in this case 1, and order/sort them in increasing order. (2 3) $2 = x_{i,1}$, $3 = y_{i,1}$
3. Check if $2<3$. If yes, then $213 < 312$ in Bruhat ✓

Ex. 2 $n=9$ Is $x = 368475912 < y = 694287531$?

1. Find $D_R(x) = \{3, 5, 7\}$
2. Take the first i digits of x and y, in this case 7, and order sort them in increasing order:

   $3456789 < 2456789$ (fails here, but keep going for example)

order first 5: $34678 < 24689$

order first 3: ~~[illegible]~~ $368 < 469$

3. Check if ~~[illegible]~~, no since $3 > 2$, so 345

$368 < 469$

Ex. 3 $n=7$ $\pi_1 = 4217653$ $\pi_2 = 5374612$

1. ~~$D_R(x)$~~ $D_R(\pi_1) = \{1,2,4,5,6\}$

2. take the first i digits of π_1 and π_2, 6, and order them in increasing order.

6: $124567 < 134567$

5: $12467 < 34567$

4: $1247 < 3457$ ~~[illegible]~~ fails here!

SO ~~π_1~~ $4217653 \not< 5374612$

(2)

Ex 4. Is $\pi_1 = 21435 < \pi_2 = 54312$?

1. $D_R(\pi_2) = \{1,3\}$
2. Sort the first 3 and then ~~3~~ ~~[illegible]~~ 1 digits of π_1 and π_2.
   $124 < 345$
   $2 < 5$
3. compare each value, it is less than for every number, so we can say $\pi_1 < \pi_2$ in Bruhat.

At the extremes, for $n = 5$, we can see that:
$e = 12345$ has no descents
$w_0 = 54321$ has $D_R(w_0) = \{1,2,3,4\}$ and maximal descents

By Picture / Intuitively $\ell(w) = \#$ inversions ~~in~~ $(w) = \{(i,j) \mid \pi(i) > \pi(j)\}$
by definition of Bruhat $v < w \Rightarrow \ell(v) < \ell(w)$

Parabolic subgroups and Quotients

For $J \subseteq S$, let W_J be the subgroup ~~A~~ of W generated by the set J.
Subgroups of (W,S) of this form are called parabolic.

Prop 2.4.1 (i) (W_J, J) is a coxeter group. ~~[illegible]~~
(ii) $\ell_J(w) = \ell(w)$, for all $w \in W_J$.
(iii) $W_I \cap W_J = W_{I \cap J}$.
(iv) $\langle W_I \cup W_J \rangle = W_{I \cup J}$.
(v) $W_I = W_J \Rightarrow I = J$.

Consequently, parabolic subgroups form a sublattice of W's subgroup lattice that is isomorphic to the boolean lattice 2^S. The coxeter diagram for (W_J, J) is obtained by removing all nodes in $S \setminus J$ and their incident ~~edges~~ edges from the diagram for (W,S).

Since W_J is finite, it has top element $w_0(J)$ = top element of W_J. Thus $w_0(\emptyset) = e$ and $w_0(S) = w_0$ (If W is finite). ~~Clearly~~ Clearly $w_0(I) \neq w_0(J)$ if $I \neq J$.

Abstract definition of a Descent Class.

Def 2.4.2 For $I \subseteq J \subseteq S$, let

$D_I^J = \{ w \in W \mid I \subseteq D_R(w) \subseteq J \}$

$W^J = D_\emptyset^{S \setminus J}$

$D_I = D_I^I$

Sets of ~~the~~ the form D_I^J are called right descent classes. W^J is a special descent class called a quotient defined $W^J = \{ w \in W \mid ws > w \ \forall s \in J \}$

Lemma 2.4.3 An element w belongs to W^J iff no reduced expression for w ends with a letter from J.

Prop 2.4.4 Let $J \subseteq S$. Then the following hold.

(i) Every $w \in W$ has a unique factorization $w = w^J \cdot w_J$ such that $w^J \in W^J$ and $w_J \in W_J$.

(ii) For this factorization, $\ell(w) = \ell(w^J) + \ell(w_J)$

Proof: (Existence) Choose $s_1 \in J$ so that $ws_1 < w$, if such s_1 exists. Continue choosing $s_i \in J$ so that $ws_1 \ldots s_i < ws_1 \ldots s_{i-1}$ as long as such s_i can be found. The process must end after at most $\ell(w)$ steps. If it ends with $w_k = ws_1 \ldots s_k$, then $w_k s > w_k$ for all $s \in J$; that is, $w_k \in W^J$. Now, let $v = s_k s_{k-1} \ldots s_1 \in W_J$. We have that $w = w_k v$, and by construction $\ell(w) = \ell(w_k) + k$.

(Uniqueness) Suppose that $w = uv = xy$ with $u, x \in W^J$ and $v, y \in W_J$. Let $u = s_1 s_2 \ldots s_k$ and $vy^{-1} = s_1' s_2' \ldots s_q'$ with the first expression reduced, $s_i \in S$, $s_j' \in J$. Then,

$$x = uvy^{-1} = s_1 s_2 \ldots s_k s_1' s_2' \ldots s_q'$$

From this, we can extract a reduced subword for x. It cannot end in some letter s_j', since $x \in W^J$. Hence, it is a subword of $s_1 s_2 \ldots s_k$, and $x \leq u$ follows. By symmetry $u = x$ and $v = y$ follows.

4

Parabolic subgroups of S_n are often called Young Subgroups.
For notational simplicity, describe only maximal parabolic subgroups and their quotients. All permutations $x \in S_n$ will be denoted in complete notation as $x = x_1 x_2 \dots x_n$ where $x_i = x(i)$. For $k \in [n-1]$, let

$$S_n^{(k)} = \{x \in S_n \mid x_1 < \dots < x_k \text{ and } x_{k+1} < \dots < x_n\}$$

From convention of letting s_i denote the adjacent transpositions $(i, i+1)$,

Lemma 2.4.7 Let $J = S \setminus \{s_k\}$. Then, $(S_n)_J = \mathrm{Stab}([k]) \cong S_k \times S_{n-k}$ and $(S_n)^J = S_n^{(k)}$.

What does this mean? The reason why Parabolic subgroups are special is that you can read the result from the coxeter graph

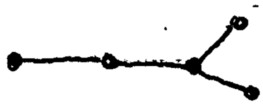
Ex S_4 let $k=2$ $(S_4)_J = \{s_1, s_3\}$

delete vertex corr to s_k and look at resulting graph

$\bullet\, S_2 \quad \bullet\, S_2 = S_2 \times S_2$

Thus $S_4 \setminus \{s_2\} = S_2 \times S_2$

Ex D_5:



$(D_5)_J = S_5$

If you understand subgroups well enough, you understand the whole group.

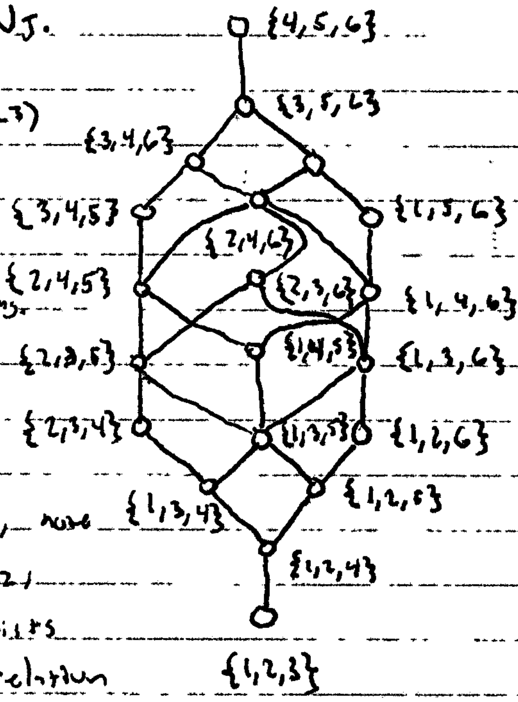
Proposition 2.4.8 For $x, y \in S_n^{(k)}$, the followings are equivalent
(i) $x \leq y$
(ii) $x_i \leq y_i$, for $1 \leq i \leq k$.
(iii) $x_i \geq y_i$, for $k+1 \leq i \leq n$.

An element $x \in S_n^{(k)}$ is determined by the set $\{x_1, x_2, \dots, x_k\}$, so we can make the identification $S_n^{(k)} \leftrightarrow \binom{[n]}{k}$. Thus proposition 2.4.8 shows that the maximal parabolic quotient $S_n^{(k)}$ under Bruhat order can be identified with the family of k-subsets of $[n]$ under the product of k-tuples.

Bruhat Order on Quotients

Much of the structure found in Bruhat order on all of W is inherited when restricting to the sub poset W^J. This can to some extent be understood as transfer of structure via the projection map defined as follows. Let $J \subseteq S$, define a mapping $P^J : W \to W^J$, by $P^J(w) = w^J$. In other words, the projection map P^J sends w to it's minimal coset representative modulo W_J.

Figure 2.7 The Bruhat Poset $S_6^{(3)}$



Proposition 2.5.1 The map P^J is order-preserving.

Proof. Suppose that $w_1 \leq w_2$ in W. We will show that $w_1^J \leq w_2^J$ by induction on $\ell(w_2)$. To begin with, note that $w_1^J \leq w_1 \leq w_2$. Hence, if $w_2^J = w_2$, we are done. If not, then there exists some $s \in J$ such that $w_2 s < w_2$. The relation $w_1^J \leq w_2$ can be lifted to $w_1^J \leq w_2 s$. By induction, $w_1^J \leq (w_2 s)^J = w_2^J$.

Corollary 2.6.2 Let $u, w \in W$. Then, $u \leq w \iff P^{S\setminus\{s\}}(u) \leq P^{S\setminus\{s\}}(w)$, $\forall s \in D_R(u)$.
If (W,S) is finite $u \leq w \iff P^{S\setminus\{s\}}(w_0 w) \leq P^{S\setminus\{s\}}(w_0 u)$, $\forall s \in S\setminus D_R(w)$

We now return to the topic of describing Bruhat order for symmetric groups.

Theorem 2.6.3 (Tableau Criterion) For $x, y \in S_n$, let $x_{i,k}$ be the i-th element in the increasing rearrangement of $x_1, x_2, \ldots, x_k$, and similarly define $y_{i,k}$. Then, the following are equivalent:
(i) $x \leq y$.
(ii) $x_{i,k} \leq y_{i,k}$, for all $k \in D_R(x)$ and $1 \leq i \leq k$.
(iii) $x_{i,k} \leq y_{i,k}$, for all $k \in [n-1]\setminus D_R(y)$ and $1 \leq i \leq k$.

Proof: Condition (ii) can be restated as saying that $P^{S\setminus\{k\}}(x) \leq P^{S\setminus\{k\}}(y)$ for all $k \in D_R(x)$. Similarly condition (iii) says that $P^{S\setminus\{k\}}(w_0 y) \leq P^{S\setminus\{k\}}(w_0 x)$ for all $k \in D_R(w_0 y)$. The result therefore follows from Corollary 2.6.2.

ex. Check whether $x = 368475912 \leq y = 694287531$, using version (iii)

1. Get $D_R(y) = \{2, 3, 5, 6, 7, 8\}$, then take $[8]\setminus D_R(y) = \{1, 4\}$
2. arrange first 4 then first 1
   $3468 \leq 2469$
   $3 \leq 6$
3. Evaluate, since $3 \not\leq 2$, $x \not\leq y$