Outline

5.4 The Möbius Function of a Poset

poset : partially ordered set $(P, \leq)$

where $\leq$ is reflexive, antisymmetric & transitive & some elements are not comparable

$\forall x, y \in P : x < y, x = y, x > y$ or not

ex] $(\mathbb{N}, |)$ $\quad 2 \nmid 3$ and $3 \nmid 2$ and $2 \neq 3$

2 & 3 not comparable

You may know the special case of this function from number theory.

For any positive integer n define $\mu(n)$ as either $-1, 0$ or 1 depending on the factorisation of n into prime factors.

$\mu(n) = 1$ if n is a square-free positive integer with even no. of prime factors $\quad$ ex. $2 \cdot 3 \cdot 5 \cdot 7 = 210$

$\mu(n) = -1$ " " with an odd no. of prime factors $\quad$ ex. $2 \cdot 3 \cdot 5 = 30$

$\mu(n) = 0$ if n has a squared prime factor $\quad$ ex. $2^2 = 4$, $24 = 2^3 \cdot 3$

whats $\mu(1)$? $= 1$ because 0 prime factors.

Now if we look at the more generalised defn of the

Möbius function of a poset this is more powerful

Let P be a locally finite poset with a $\hat{0}$. The Möbius function of P is a map $\mu : P \to \mathbb{Z}$ defined inductively by

$\mu : P \to \mathbb{Z}$ integers

$$(5.4) \quad \mu(x) = \begin{cases} 1 & \text{if } x = \hat{0} \quad \text{minimum} \\ -\sum_{y < x} \mu(y) & \text{otherwise} \end{cases}$$

P is locally finite so the no. of summands is finite so that μ is well-defined.

We can move terms to the LHS of the eqn to get the equiv. defn.

$$\left[\ \delta_{\hat{0},x} = \begin{cases} 0, & x \neq \hat{0} \\ 1, & x = \hat{0} \end{cases} \quad \text{Kronecker delta}\ \right]$$

Assume $\mu(x) \neq \hat{0}$

Then $\mu(x) = -\sum_{y<x} \mu(y)$

$$\sum_{y \leq x} \mu(y) = \mu(x) + \sum_{y<x} \mu(y) = 0 = \delta_{\hat{0},x}, \quad x \neq \hat{0}$$

If $x = \hat{0}$ $\quad \mu(x) = 1 = \delta_{\hat{0},\hat{0}}$

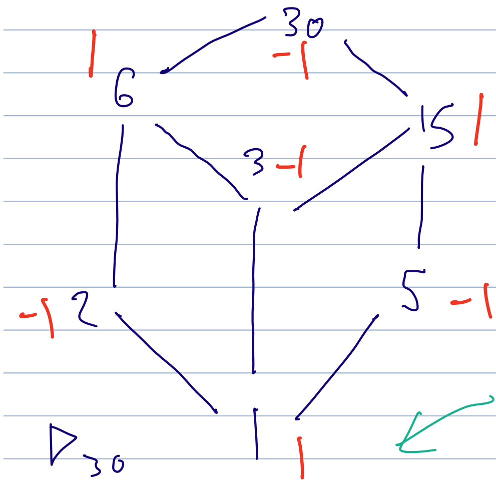
We'll use this later

$$\sum_{y \leq x} \mu(y) = \delta_{\hat{0},x} \qquad (5.5)$$

Hasse diagrams

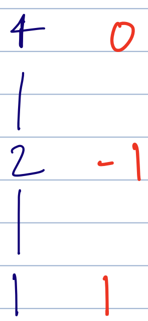
number theory is a specific example of more general definition

D_{30}

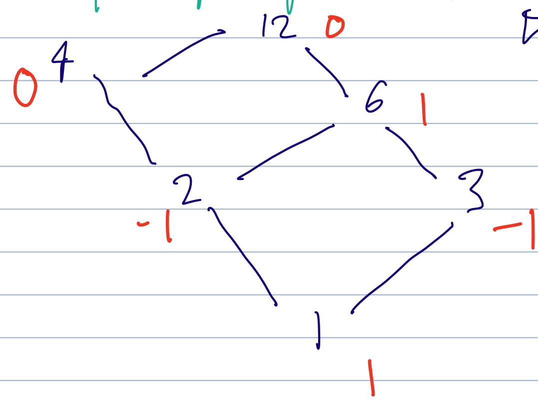


ask class for μ of each number

D_4

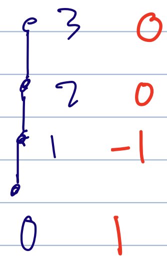


D_{12}



C_3

chain



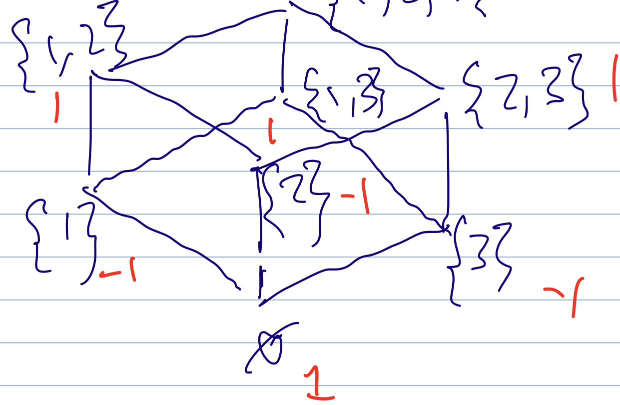
Prop 5.4.1 In C_n we have

$$\mu(i) = \begin{cases} 1 & \text{if } i=0 \\ -1 & \text{if } i=1 \\ 0 & \text{otherwise} \end{cases}$$

So $\mu(C_n) = 1, -1,$ or 0 depending on whether $n=0, 1$ or $n \geq 2$.

B_3 Power set of a set ordered by containment

$(P(S), \subseteq)$



Prop. 5.4.2 If $S \in B_n$ then

$$\mu(S) = (-1)^{\# S}$$

so $\mu(B_n) = (-1)^n$

Proof: show that $(-1)^{\# S}$ satisfies (5.5)

$$\sum_{y \leq x} \mu(y) = \delta_{\hat{0}, x}$$

Suppose $T \in B_n$ and let $\# T = k$ then

$$\sum_{S \subseteq T} (-1)^{\# S} = \sum_{i=0}^{k} \sum_{S \in \binom{T}{i}} (-1)^i = \sum_{i=0}^{k} \binom{k}{i} (-1)^i = \delta_{0,k} = \delta_{\phi, T}$$

$T = \{1, \dots, k\}$ $\quad S \subseteq T$ $\quad \# S = i$

any subset of T

$S \in \binom{T}{i}$ All of subsets of T with size i

S can be thought of as taking T and take i elements from it

there are $\binom{\# T}{i}$ which is k many such sets

$$\sum_{i=0}^{k} \binom{k}{i} (-1)^i = \sum_{i=0}^{k} \binom{k}{i} 1^{k-i} (-1)^i = (1 + (-1))^k = 0^k$$

Unless $T = \emptyset$ then we can see equal to 0

If T $\emptyset$ $\mu(\emptyset) = (-1)^0 = 1 = \delta_{\emptyset,\emptyset}$

Theorem 5.4.3

Theorem 5.4.3. *Let P be a locally finite poset with $\hat{0}$ and let $f : P \to Q$ be in isomorphism. Then for all $x \in P$ we have*

$$\mu_P(x) = \mu_Q(f(x)).$$

Two posets being isomorphic means they have the same Hasse diagram & so the Mobius function will be the same.

Theorem 5.4.4. *Let P and Q be locally finite posets containing $\hat{0}_P$ and $\hat{0}_Q$, respectively. Then for all $s \in P$ and $x \in Q$ we have*

$$\mu_{P\times Q}(s,x) = \mu_P(s)\mu_Q(x).$$

Proof. It suffices to show that the right-hand side of the displayed equation satisfies (5.5). But given $(s,x) \in P \times Q$, we have

$$\sum_{(t,y)\leq(s,x)} \mu_P(t)\mu_Q(y) = \sum_{t\leq s}\mu_P(t) \sum_{y\leq x}\mu_Q(y) = \delta_{\hat{0}_P,s}\delta_{\hat{0}_Q,x} = \delta_{(\hat{0}_P,\hat{0}_Q),(s,x)}$$

as desired.

def. Kronecker delta

If $t \leq s$, then for any $y \leq x$ $(t,y) \leq (s,x)$

also if $y \leq x$, then for any $t \leq s$ $(t,y) \leq (s,x)$

ordered pairs are equal iff both words are equal

$$\delta_{a,b} \cdot \delta_{c,d} = \delta_{(a,c),(b,d)}$$

get zero unless $a = b$ and $c = d$

then $(a,c) = (b,d)$

We can now compute the Möbius function of the divisor lattice.

Proposition 5.4.5. *The Möbius function of D_n is*

$$\mu(d) = \begin{cases} (-1)^m & \text{if } d \text{ is a product of } m \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases} \tag{5.7}$$

This is the number theory example mentioned earlier

Poset : partially ordered set $(P, \leq)$

where $\leq$ is reflexive, antisymmetric & transitive and some elements are not comparable.

$\forall x, y \in P$: $x < y$, $x = y$, $x > y$ or more of these things

ex $(\mathbb{Z}, |)$ $\quad 2 \nmid 3$ and $3 \nmid 2$ and $2 \neq 3$

2 and 3 are not comparable

Hasse diagrams p. 141,

Proposition 5.4.2. *If $S \in B_n$, then*

$$\mu(S) = (-1)^{\#S}. \tag{5.6}$$

So $\mu(B_n) = (-1)^n$.

Proof. It will suffice to show that the function $(-1)^{\#S}$ satisfies (5.5) since that equation uniquely defines μ. So suppose $T \in B_n$ and let $\#T = k$. Then, using Theorem 1.3.3(d),

$$\sum_{S \subseteq T} (-1)^{\#S} = \sum_{i=0}^{k} \sum_{S \in \binom{T}{i}} (-1)^i = \sum_{i=0}^{k} \binom{k}{i} (-1)^i = \delta_{0,k} = \delta_{\emptyset,T}$$

powerset of T

which is the desired equality. $\square$

all subsets w. size i

If $\#T = k$ and $0 \leq j \leq k$, then no. subsets of size j is $\binom{k}{j}$ number of ways to uniquely take j elements from k and form a subset.

$\binom{T}{j}$ = subsets of T with size j
(choosing j elements from T)

Let $(B_n, \subseteq)$ be the poset of $P(\{1, \ldots n\})$ with subset containment as our inequality

$k = 1, \ldots, n \qquad = T = \leq . \quad k \leq n$

$$\sum_{j=0}^{k} \binom{k}{j} (-1)^j$$

See note.

minimal element

Theorem 5.4.3. *Let P be a locally finite poset with $\hat{0}$ and let $f : P \to Q$ be ~~an~~ a isomorphism. Then for all $x \in P$ we have*

$$\mu_P(x) = \mu_Q(f(x)).$$

mobius function for $P(x)$ = mobius function of Q of $f(x)$ which is the image of x.

Proof two posets etc. from Caillme

5.4.4. Let P & Q

Theorem 5.4.4. *Let P and Q be locally finite posets containing $\hat{0}_P$ and $\hat{0}_Q$, respectively. Then for all $s \in P$ and $x \in Q$ we have*

$$\mu_{P \times Q}(s, x) = \mu_P(s)\mu_Q(x).$$

$P \times Q$

Proof

Our third method to produce new posets from old ones is via products. Given two (not necessarily disjoint) posets $(P, \leq_P)$ and $(Q, \leq_Q)$, their (*direct or Cartesian*) *product* has underlying set

$$P \times Q = \{(x, y) \mid x \in P, y \in Q\}$$

together with the partial order

$$(x, y) \leq_{P \times Q} (x', y') \text{ if } x \leq_P x' \text{ and } y \leq_Q y'.$$

Proof. It suffices to show that the right-hand side of the displayed equation satisfies (5.5). But given $(s,x) \in P \times Q$, we have

$$\sum_{(t,y)\le(s,x)} \mu_P(t)\mu_Q(y) = \sum_{t\le s}\mu_P(t) \sum_{y\le x}\mu_Q(y) = \delta_{\hat{0}_P,s}\delta_{\hat{0}_Q,x} = \delta_{(\hat{0}_P,\hat{0}_Q),(s,x)}$$

as desired. □

$\hat{0}_{P\times Q} = (\hat{0}_p, \hat{0}_q)$

Kronecker delta in both factors

choose some pair (s,x) in $P \times Q$
Think of all pairs (t,y) less than or equal to (s,x)

If $t \le s$, then for any $y \le x$ $(t,y) \le (s,x)$

also. if $y \le x$, then for any $t \le s$ $(t,y) \le (s,x)$

only way getting a one if

ordered pairs are equal iff. both coordinates are equal

We can now compute the Möbius function of the divisor lattice.

Proposition 5.4.5. *The Möbius function of D_n is*

(5.7) $$\mu(d) = \begin{cases} (-1)^m & \text{if } d \text{ is a product of } m \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

square free

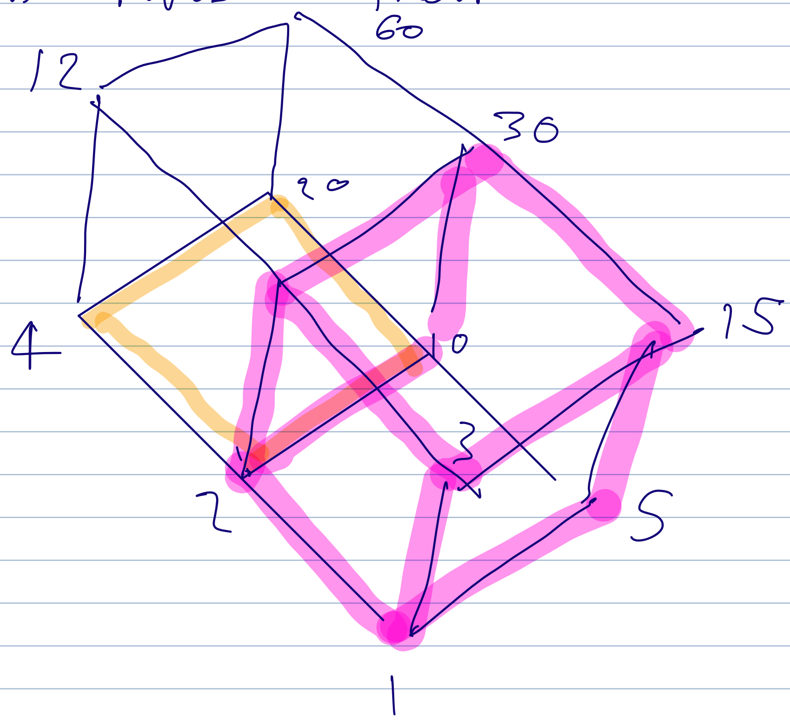
not square free

number theory example

$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, $p_i \neq p_j$

divisibility is the motivating example for the Möbius function

Mobius Inversion Theorem



$[2, 20]$
$= \{2, 4, 10, 20\}$

$\hat{0}_{[2,20]}$

Then the corresponding *closed interval* is

$[x, y] = \{z \in P \mid x \leq z \leq y\}.$

eg. $[2, 5] = \phi$

$$\mu(2,20) = -\sum_{\substack{2|y \\ y|20}} \mu(2,y) \qquad y \neq 20$$

$$= -[\mu(2,4) + \mu(2,10) + \mu(2,2)]$$

$$= -[-1 + -1 + 1]$$

$$= 1$$

Skip if not enough

5.5.7. inc./excl.

is an application of the mobius function using something v. general

5.5. The Möbius Inversion Theorem

In this section we will prove the Möbius Inversion Theorem, which is a very general method for inverting sums over posets P. In fact, we will show that special cases of this result include the Fundamental Theorem of the Difference Calculus ($P = C_n$), the Principle of Inclusion and Exclusion ($P = B_n$), and the Möbius Inversion Theorem in number theory ($P = D_n$). A useful perspective will be to consider a certain algebra associated with P called the incidence algebra and which permits linear algebra techniques to be employed.

Our first step will be to generalize the Möbius function to a map having two arguments. Let P be a locally finite poset and let $\mathrm{Int}(P)$ be the set of closed intervals of P. Note that every $[x,z] \in \mathrm{Int}(P)$ has a minimum element; namely $\hat{0}_{[x,y]} = x$. The *Möbius function* of P is the map $\mu : \mathrm{Int}(P) \to \mathbb{Z}$ defined inductively on $[x,z]$ by

5.12 using only Int(P) that have the same minimal element.

5.13 analogous to what we had in 5.5

(5.5) $$\sum_{y \le x} \mu(y) = \delta_{\hat{0},x}$$

$$\mu(x,z) = \mu_{[x,z]}(z)$$

(5.13) $$\sum_{x \le y \le z} \mu(x,y) = \delta_{x,z}.$$

Theorem 5.5.5 (Möbius Inversion Theorem). *Let P be a finite poset, let V be a real vector space, and let $f, g : P \to V$ be two functions.*

(a) *We have*

$$f(x) = \sum_{y \ge x} g(y) \text{ for all } x \in P \iff g(x) = \sum_{y \ge x} \mu(x,y) f(y) \text{ for all } x \in P.$$

(b) *We have*

$$f(x) = \sum_{y \le x} g(y) \text{ for all } x \in P \iff g(x) = \sum_{y \le x} \mu(y,x) f(y) \text{ for all } x \in P.$$

$V = \mathbb{R}$ for ease of understanding

Proof. We will prove (a), leaving (b) as an exercise. In fact, we will give two proofs of (a), one working directly with the elements of $\mathcal{I}(P)$ and one using linear algebra.

Let us assume that $f(x) = \sum_{y \geq x} g(y)$ for all $x \in P$. Plugging this into summation involving μ and using (5.13) yields

$$\begin{aligned}
\sum_{y \geq x} \mu(x,y) f(y) &= \sum_{y \geq x} \mu(x,y) \sum_{z \geq y} g(z) \\
&= \sum_{z \geq x} g(z) \sum_{x \leq y \leq z} \mu(x,y) \\
&= \sum_{z \geq x} g(z) \delta_{x,z} \\
&= g(x).
\end{aligned}$$

we have function g &
make f using this formula

Extend this to vectors by using a
matrix

Our first application will be to the theory of finite differences, which is a discrete analogue of the calculus. A function $f : \mathbb{N} \to \mathbb{R}$ has as *(forward) difference* the function $\Delta f : \mathbb{N} \to \mathbb{R}$ defined by

$$\Delta f(n) = f(n+1) - f(n).$$

This corresponds to differentiation. Indeed, the derivative of $f : \mathbb{R} \to \mathbb{R}$ is

$$f'(x) = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

and at $\epsilon = 1$ the function inside the limit is just $f(x+1) - f(x)$. For example, if $f(n) = n^2$, then $\Delta f(n) = (n+1)^2 - n^2 = 2n + 1$ which bears a strong resemblance to $(x^2)' = 2x$. There is also a version of the definite integral in this context. The *definite summation* of $f : \mathbb{N} \to \mathbb{R}$ is the function $Sf : \mathbb{N} \to \mathbb{R}$ where

$$Sf(n) = \sum_{i=0}^{n} f(i).$$

The analogue of the Fundamental Theorem of Calculus is as follows. It will be convenient to extend the domain of any $f : \mathbb{N} \to \mathbb{R}$ to $\mathbb{Z}$ by letting $f(i) = 0$ for $i < 0$.

Theorem 5.5.6 (Fundamental Theorem of Difference Calculus). *Given two function $f, g : \mathbb{N} \to \mathbb{R}$, we have*

$$f(n) = Sg(n) \text{ for all } n \geq 0 \iff g(n) = \Delta f(n-1) \text{ for all } n \geq 0.$$

Proof. It is easy to compute that in the chain C_n we have

$$\mu(i,n) = \begin{cases} 1 & \text{if } i = n, \\ -1 & \text{if } i = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Now for all $n \geq 0$, the first condition in the theorem can be translated as

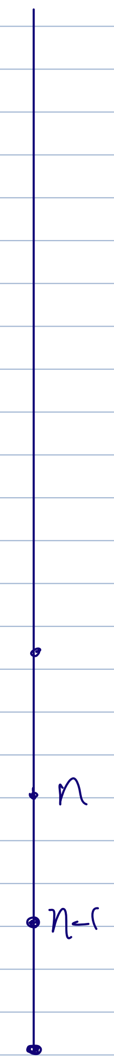
$$f(n) = Sg(n) = \sum_{i=0}^{n} g(i) = \sum_{i \leq n} g(i)$$

where the inequality indexing the last summation is taking place in C_n. Using Theorem 5.5.5(b) and the Möbius values in C_n above, this is equivalent to

$$g(n) = \sum_{i \leq n} \mu(i,n) f(i) = (1)f(n) + (-1)f(n-1) = \Delta f(n-1)$$

for all $n \geq 0$. □

It turns out that the Principle of Inclusion and Exclusion is just the Möbius Inversion Theorem applied to the poset B_n. We restate it here for ease of reference.



$\mu(n,n) = 1$

$\mu(n-1,n) = -\mu(n-1,n-1) = -1$

$\mu(n-2,n) = -[\mu(n-2,n-1) + \mu(n-2,n-2)]$

$= -[-1 + 1] = 0$

Theorem 5.5.7 prove using the mobius inversion formula

Theorem 5.5.8

Theorem 5.5.8. *Given two functions $f, g : \mathbb{P} \to \mathbb{R}$, we have*

$$f(n) = \sum_{d|n} g(d) \text{ for all } n \in \mathbb{P} \iff g(n) = \sum_{d|n} \mu(d) f(n/d) \text{ for all } n \in \mathbb{P}.$$

apply number theory function from before.