An Overview on Generating Functions

Dawson Franz [1]

Abstract: In this lecture we will discuss generating functions, which are used to represent and analyze the solution to an enumeration problem, often in a simpler form than the sequence that defines the problem. These functions are a powerful "bookkeeping tooländ we will show how they can be used to solve generalized recurrence relations and to find a generic term of a recurring sequence with k terms.

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1 Terminology Review

This lecture will mainly focus on how to solve and succinctly represent the answer to *enumeration problems* using something called a generating function. But first, what exactly is an enumeration problem? It is figuring out how many objects of size n fit a specific definition. Some examples of enumeration problems include the following:

1. How many permutations are there of the set $\{1, 2, ..., n\}$?
   a. There are $\mathbf{n}!$ permutations
2. How many binary sequences, or strings, are there of length n?
   a. There are $\mathbf{2^n}$ unique strings ($0...00, 0...11, ..., 1...11$ where each is length n).

Before starting this lecture, here's a few notations and definitions that are helpful to remember as we see more examples:

- The **binomial theorem** (which expands something raised to a finite power) states:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

- A **class** $\mathbb{A}$ is a collection of sets $\{\mathbb{A}_n\}$ indexed by natural numbers $n \in \mathbb{N}$. For example, if $\mathbb{A}$ is a class of permutations, $\mathbb{A}_n$ is the class of permutations of size n.
- $\mathbf{a_n}$, which is the number of objects of size n, represents a sequence of numbers.

[1] Columbia University, Mathematics, New York, New York, drf2138@columbia.edu

2 Defining Generating Functions

A **generating function** is another way to represent a sequence of numbers or a "bookkeeping tool."This form is often simpler than the sequence itself.

Def. 1: Let $(a_n)_{n\geq 0}$. This sequence's generating function is the following series:

$$A(x) = \sum_{n\geq 0} a_n x^n = a_0 x^0 + a_1 x^1 + ... + a_n x^n$$

This also defines the generating function of an enumerable class $\mathbb{A}$ with a_n objects of size n in the class.

2.1 Example: Binomial Theorem

Let $a_n = \binom{k}{n}$ for $n \leq k$ and $a_n = 0$ for $n > k$.

Through the binomial theorem, the sum for A(x) can be represented as

$$A(x) = \sum_{n\geq 0} a_n x^n = \sum_{n\geq 0} \binom{k}{n} x^n = (1 + x)^k$$

So the *generating function* of $A(x)$ is $(1+x)^k$, where each element is a subset of $\{1, 2, ..., k\}$ with the size n equal to the number of elements.

2.2 Example 2: Coin Tossing

A two-sided coin, with probability $p \geq 0$ to land on heads and $q = 1 - p$ to land on tails, is thrown k times. Let a_n be the probability of seeing exactly n heads, or $a_n = \binom{k}{n} q^{k-n} p^n$. Using the binomial theorem, the *generating function* for the sequence is

$$A(x) = \sum_{n\geq 0} a_n x^n = \sum_{n\geq 0} \binom{k}{n} q^{k-n} p^n = (q + px)^k$$

which is equivalent to

$$\underbrace{(q + px)(q + px)...(q + px)}_{\text{k times}}$$

(The generating function $A(x) = (q + px)^k$, when multiplied out, has 2^k terms, with combinations of coefficients q (the number of tails) and p (the number of heads) that correspond to each possible sequence of coin flips of size n.)

3 Multiplication of Generating Functions

Taking the Cartesian Product of two classes is equivalent to multiplying their generating functions. The dice example below Theorem 1 is an intuitive way of seeing this.

Theorem 1. Let $\mathbb{A}$ and $\mathbb{B}$ be classes with generating functions $A(x)$ and $B(x)$. Then the class $\mathbb{C} = \mathbb{A} \times \mathbb{B}$ has the generating function $C(x) = A(x)B(x)$.

Proof. Let c_n be the number of objects of size n in the Cartesian product $\mathbb{C} = \mathbb{A} \times \mathbb{B}$. To make each object $c = (a, b)$, we pick an object $a \epsilon \mathbb{A}$ of size $k \leq n$ and an object $b \epsilon \mathbb{B}$ of size $n - k$. So we have a total size of $n = k + (n - k)$ for each object $c = (a, b)$:

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Now, looking at the individual generating functions of $A(x) = \sum_{k \geq 0} a_k x^k$ and $B(x) = \sum_{k \geq 0} b_k x^k$, their product is

$$A(x)B(x) = \left(\sum_{k \geq 0} a_k x^k \right) \times \left(\sum_{k \geq 0} b_k x^k \right)$$

To get an element with x^n from this product (which is needed to show $C(x) = \sum_{n \geq 0} c_n x^n = A(x)B(x)$), we need the exponents from each a and b to multiply together to n. So, you can multiply each $a_k x^k$ for $k \leq n$ from $A(x)$ by $b_{n-k} x^{n-k}$ from $B(x)$:

$$A(x)B(x) = \sum_{n \geq 0} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n = \sum_{n \geq 0} c_n x^n = C(x)$$

Which completes the proof. □

3.1 Example: Multiplying Generating Functions of Two Dice

Now as an example of the above theorem, imagine we have two dice: a 6-sided die (numbered 1 to 6) and a 8-sided die (numbered 1 to 8). We roll both dice to get a sum and want to know how many ways c_n we can get each sum n. The generating function for the sum, $C(x) = \sum_{n\geq 0} c_n x^n$, is given by

$$C(x) = \underbrace{(x + x^2 + x^3 + x^4 + x^5 + x^6)}_{\text{possible outcomes of the first die}} \times \underbrace{(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8)}_{\text{possible outcomes of the second die}}$$

with the result of this multiplication being

$$C(x) = (x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 6x^8 + 6x^9 + 5x^{10} + 4x^{11} + 3x^{12} + 2x^{13} + x^{14})$$

where each exponent n is the combined value of the two dice (the size), and each coefficient c_n is the number of possible outcomes with that size.

(If we consider a die of i sides as A_i, the above example can also be written as $C(x) = A_6 A_8$.)

Notation Note: If we multiply a class by itself multiple times, we use the notation

$$\mathbb{A}^k = \underbrace{\mathbb{A} \times \mathbb{A} \times \ldots \times \mathbb{A}}_{\text{k times}}$$

By theorem 1, the generating function for this class is $A(x)^k = \underbrace{A(x) \times A(x) \times \ldots \times A(x)}_{\text{k times}}$.

For example, the generating function for the sum of 5 six-sided dice is

$$C(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^5$$

4 Dots and Dashes Example

Suppose we need to send a message using only dots and dashes (like in Morse code), where a dot • takes 1 time unit to send, and a dash — takes 2 time units. How many different messages can we send in n time units? We can let f_n represent the number of ways to send a message of size n, where n is the total number of time units in the message. We can make a table with some of the first messages in the sequence to see if we can find a pattern:

n	1	2	3	4	5	...
f_n	1	2	3	5	8	...
messages of time n	•	•• —	••• — • • —	•••• — •• • — • •• — — —	••••• — ••• • — •• •• — • ••• — — — • — • — • — —	

So for any f_n, we can try to define this relationship recursively. We know that the last symbol has to be either a dot or a dash. If the last symbol is a dot, then we have $f_n = f_{n-1} +$ dot, otherwise if it's a dash, then we have $f_n = f_{n-2} +$ dash. Observing the pattern a bit closer, we can also clearly see it's similar to the Fibonacci sequence:

$$f_n = f_{n-1} + f_{n-2}$$

We can write this in summation notation to connect it to generating functions:

$$F(x) = \sum_{j=0}^{\infty} f_j x^j$$

But notice that this definition includes f_0, the empty message with $n = 0$. So if we wrote out this series, we'd get

$$F(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + \ldots$$

and since we need to define f_n in terms of the previous two elements, we want to also define the generating function $F(x)$ in terms of itself to try to solve for a closed form. So let's shift the terms by multiplying by factors of x:

$$\begin{aligned} xF(x) &= \quad x + x^2 + 2x^3 + 3x^4 + \ldots \\ x^2F(x) &= \qquad 1x^2 + 1x^3 + 2x^4 + \ldots \end{aligned}$$

And adding together $xF(x) + x^2F(x)$ gives us $x + 2x^2 + 3x^3 + 5x^4 + \ldots = F(x) - 1$, showing that the recursive sequence needs another initial element to account for f_0. Therefore, solving for the recursive generating function, we get

$$F(x) = 1 + xF(x) + x^2F(x) = \frac{1}{1 - x - x^2}$$

To get more information on the coefficients f_n of the generating function, we can factor the polynomial we get and convert it to a sum of simpler rational functions using partial fraction decomposition. So first, we get

$$1 - x - x^2 = (1 - r_1 x)(1 - r_2 x)$$

where $r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2}$ are the inverses of the roots of the function, and we can break apart this fraction using these roots:

$$F(x) = \frac{a}{1 - r_1 x} + \frac{b}{1 - r_2 x} \quad \text{where} \quad a = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right), \; b = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)$$

The reason why we're factoring out the roots into simpler fractions is so that we can get the generating function in the form of sums of geometric series $\frac{1}{1-\alpha x}$. The Taylor series of a geometric function is

$$\frac{1}{1 - \alpha x} = 1 + \alpha x + \alpha^2 x^2 + \alpha^3 x^3 + \ldots$$

Now we can write out $F(x)$ in terms of simple geometric series to try to solve for f_n, so if we expand the series we get the following, with the *polynomials now only in the numerator*:

$$F(x) = a(1 + r_1 x + r_1^2 x^2 + r_1^3 x^3 + \ldots) + b(1 + r_2 x + r_2^2 x^2 + r_2^3 x^3 + \ldots)$$

And because we know that $a = \frac{1}{\sqrt{5}}(r_1), \; b = -\frac{1}{\sqrt{5}}(r_2)$, then

$$F(x) = \frac{1}{\sqrt{5}}(r_1 + r_1^2 x + r_1^3 x^2 + r_1^4 x^3 + \ldots) - \frac{1}{\sqrt{5}}(r_2 + r_2^2 x + r_2^3 x^2 + r_2^4 x^3 + \ldots)$$

Which clearly shows us the form of f_n, the nth Fibonacci number. So writing down this pattern by combining the terms for each x^n monomial, we've solved for f_n, which is the coefficient of each x^n in $F(x)$:

$$f_n = \frac{1}{\sqrt{5}}\left(r_1^{n+1} - r_2^{n+1}\right) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$$

5 Generalized Recurrence Relations

Looking at the Dots and Dashes example, we can start to generalize the behavior of recurrence equations of k terms using generating functions. So, if we gave some recurrence relationship in terms of f_n, like

$$f_n = \alpha f_{n-1} + \beta f_{n-2} + \gamma f_{n-3} + \ldots$$

we can start by writing the generating function in terms of f_j and x^j:

$$F(x) = \sum_{j=0}^{\infty} f_j x^j$$

Then we can write $F(x)$ in terms of itself including shifts (up to x^k), coefficients $(\alpha, \beta, \gamma, \ldots)$, and $p(x)$, where $p(x)$ is defined as the initial terms that make the recurrence equation work (e.g. $p(x) = f_0 = 1$ in the above example).

$$F(x) = \alpha x F(x) + \beta x^2 F(x) + \gamma x^3 F(x) + \ldots + p(x)$$

The additional term $p(x)$ can have a maximum degree of $k - 1$ if the recurrence equation has k terms (since it accounts for the initial value–for example, if the recurrence equation had 3 terms, the maximum degree of $p(x)$ would be 2).

$$F(x) = \frac{p(x)}{\alpha x + \beta x^2 + \gamma x^3 + \ldots}$$

Like in the previous example, we can try to factor the denominator to get simple fractions in geometric form (letting some online tool find our roots $r_1, r_2, \ldots, r_k$ for us):

$$1 - \alpha x - \beta x^2 - \gamma x^3 - \ldots = (1 - r_1 x)(1 - r_2 x)(1 - r_3 x)\ldots$$

(Do note that since the fraction is in the denominator, these roots are actually the zeros of the inverse of the polynomial. For example, for k=3, then we have $y^3 - \alpha y^2 - \beta y - \gamma = 0$ with $y = \frac{1}{x}$). We then use partial fraction decomposition to get $F(x)$:

$$F(x) = \frac{a}{1 - r_1 x} + \frac{b}{1 - r_2 x} + \frac{c}{1 - r_3 x} + \ldots$$

So just like in the Dots and Dashes example, we look at the generating function $F(x)$, then perform a series expansion on each of these geometric equations, and finally add up the coefficients for each x^n in

$$F(x) = \sum_{n=0}^{\infty} f_n x^n$$

. Looking at the resulting pattern from each of these terms gives us f_n, a generic term of the recurring sequence, in terms of n, the constants $a, b, c, ...$, and the roots $r_1, r_2, r_3, ...$:

$$f_n = ar_1^n + br_2^n + cr_3^n + ...$$

So now, by using generating functions, we have a closed form of the general recurrence equation $f_n = \alpha f_{n-1} + \beta f_{n-2} + \gamma f_{n-3} + ...$ in terms of the roots of $\frac{p(x)}{\alpha x+\beta x^2+\gamma x^3+...}$.

6 Additional Notes

The union of classes is equivalent to the addition of their generating functions.

For sequences, we can represent their generating functions as

$$F(x) = \sum_{k\geq 0} A(x)^k = \frac{p(x)}{1 - A(x)}$$

where the set of finite sequences of elements of $\mathbb{A}$ is defined as $\mathbb{C} = Seq(\mathbb{A}) = \cup_{k\geq 0}\mathbb{A}^k$, so if $\mathbb{A} = \{0, 1\}$, then

$$\mathbb{A}^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$$

Generating functions can also help us express how many ways there are to give change for n cents given some amount of coins (see the cited article, section 3).

7 Bibliography

M. Goemans. Generating Functions. https://math.mit.edu/ goemans/18310S15/generatingfunction-notes.pdf. March 1, 2015.