Catalan Numbers & Fully Commutative Elements

Saturday, November 12, 2022 8:35 PM

INTRO

Def: Catalan Numbers

- A sequence of natural numbers that occur in various counting problems

Remark: we are already familiar with several other infinite sequences of numbers, e.g.

- the counting numbers, {1, 2, 3, 4, … }
- Perfect squares, {1, 4, 9, 16, … }
- powers of 2, {1, 2, 4, 8, 16, … } from 2^0, 2^1, 2^2, 2^3, …
- the Fibonacci sequence, {1, 1, 2, 3, 5, 8, 13, … } where $x_n = x_{n-1} + x_{n-2}$

  When talking about the catalan numbers, we're just talking about another sequence that comes up in questions such as:

Remark: to contextualize this talk, some examples of questions that are answered by the catalan numbers:

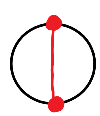
- triangulations of a convex n + 2-gon
- binary trees with n + 1 unlabeled leaves
- Dyck Paths
- well-matched expressions of n pairs of parentheses
  I'm not going to elaborate on these examples yet, they're just to give you some context for the kinds of problems that are solved by the catalan numbers

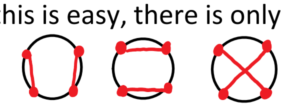
Remark: recall the formula for the choice of k things from a set of n things without replacement and where order does not matter is given by: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

The Catalan Numbers are given by:

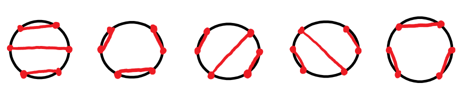
$$c_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n} = \binom{2n}{n} - \binom{2n}{n-1} = 1, 1, 2, 5, 14, 42, 132, \ldots$$

First, to get acquainted with catalan numbers, consider a round table with some number of people sitting around it. We want everyone to simultaneously shake hands with one other person, with the condition that no one's arms can cross and no one is left out. Then we know n must be an even number. So for some number of pairs of people, n, how many possible ways can this happen?

Let's start with n=1. this is easy, there is only one way. 

What about for n=2? 

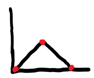
Notice there are only 2 options, one person can either shake hands with the person on their left, or the person on their right. Note, someone can't shake hands with the person across from them, or else arms would cross and that is against the rules.

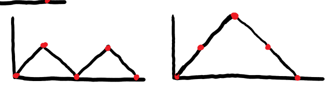
What about for n=3?
There are five options: 

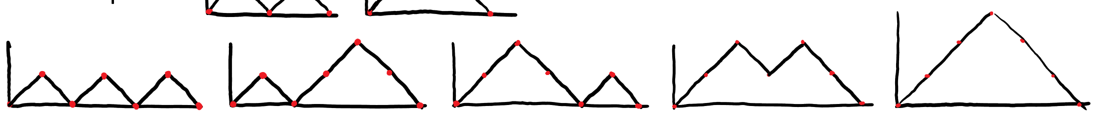
Let's move to another problem. Say we want to draw a mountain range with some number of brush strokes. The rules are:

1. Our strokes must either be up strokes or down strokes
2. We must use the same number of up strokes as down strokes
3. We must start on the horizon and end on the horizon
4. We cannot go below the horizon --> our first stroke must be an up stroke

These mountain ranges are called "Dyck Paths" after German mathematician Walther Van Dyck. Our question is, for some n, where n=number of up strokes, how many different mountain ranges can we draw?

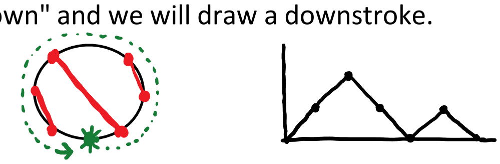
Let's look at n = 1. there is only one option: 

What about for n = 2? there are two options: 

For n=3, there are 5 options: 

Notice, for the same n, we get the same result for both problems. It turns out that these numbers show up as the answers to many, many problems. They are the Catalan numbers, given by the equation I wrote earlier: $C_n = \frac{1}{n+1}\binom{2n}{n}$

Since the answer to both of the questions we asked are the catalan numbers, this suggests that these questions must be related in some way. In fact they are. Let's see how we can take one of our handshake tables and recover from it one of our dyck paths. Let's take this table for example. Imagine there is a little beetle on the table who will move around it in a counter clockwise direction. When she encounters a handshake pair for the first time, she will shout out the word "up" and we will draw an upstroke. When she encounters a handshake pair for the second time, she will shout out "down" and we will draw a downstroke.



Thus, we recovered one of our dyck paths from one of our handshake tables. We can actually do the reverse too, and find a handshake table from one of our dyck paths. This illustrates a bijection between these two questions, and we can extend this idea to understand that there is a bijection between all questions answered by the catalan numbers (of which there are over 200). Pretty cool!

So, if we wanted to find the answer to any question solved by the catalan numbers, say for n=4, we could solve the equation:

$C_n = \frac{1}{n+1}\binom{2n}{n} \rightarrow \left(\frac{1}{n+1}\right) * \left(\frac{2n!}{n!(2n-n)!}\right) \rightarrow \frac{(2n)!}{(n+1)!n!} \rightarrow \frac{(2*4)!}{(4+1)!4!} \rightarrow \frac{8!}{5!4!} \rightarrow \frac{8*7*6*5*4*3*2*1}{(5*4*3*2*1)(4*3*2*1)} \rightarrow \frac{8*7*6*5}{5*4*3*2*1} \rightarrow \frac{8*7}{4} \rightarrow \frac{56}{4} = 14$

Then there are 14 ways that 4 pairs of people could shake hands around a table and 14 ways to draw a Dyck path with 4 upstrokes and 4 downstrokes, etc.

Now we can move on to a slightly more complicated problem.

Consider 3 pairs of parentheses. In how many ways can we validly arrange them? One way is like this: ()()()
An invalid way to arrange them would be like this:))()((

Let's consider what makes an arrangement of parentheses valid:
1. There can be no point when reading from left to right in which more sets of parentheses are closed than opened
2. Every pair of parentheses that is opened must be closed

Example: to check if this set of parentheses is valid, we can imagine every open parenthesis as +1 and every close parenthesis as -1
- If we sum the numbers one at a time from left to right, at no point should the partial sum be negative and the total sum must equal 0

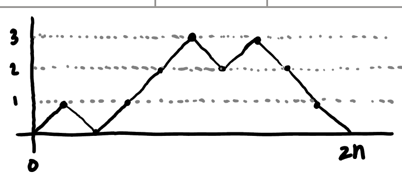
partial sum: 1 0 1 2 3 2 3 2 1 0 ← total sum = 0
+1 -1 +1 +1 +1 -1 +1 -1 -1 -1 ← never negative
( ) ((( ) ( )))

- So we see that this set of pairs of parenthesis is valid

We can represent this problem as a Dyck path by representing every open parenthesis as an up stroke and every close parenthesis as a down stroke.
- Rules:

There can be no point when reading from left to right in which more sets of parentheses are closed than opened	==	not allowed to go below horizon
Every pair of parentheses that is opened must be closed	==	path must end at 2n

()((()())) ⇒



Notice that the line never dips below the horizontal line
Also notice that there are 5 up strokes and 5 down strokes so number of up strokes and down strokes is equal

Then, this mountain meets the criteria for a Dyck path, so we can conclude that the set of parenthesis is valid. Additionally, because the number of up strokes and down strokes is 5, that tells us there are 5 pairs of parentheses.

This is cool because looking at the parenthesis, it can be hard to tell. But, because the pairs of parentheses problem is answered by the catalan numbers, there is a bijection between this problem and the Dyck path problem, so we can represent the parentheses problem with a Dyck path which is much easier to read.

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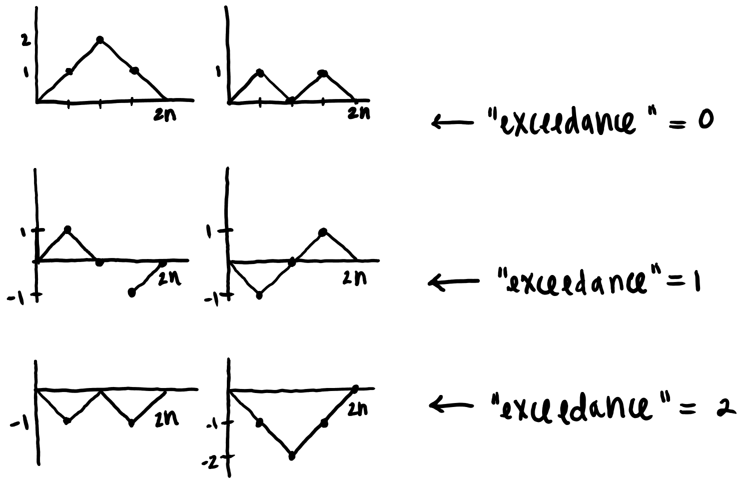
For some n, we would like to know how many valid paths there are to 2n (i.e. that follow the Dyck path rules). This is equivalent to asking how many valid pairs of n parentheses there are.

We will approach this in a combinatorial way by considering all possible paths to 2n, including the ones that violate the Dyck path rules.

Let's consider all the possible paths for n=2.



There are several things to note here:

1. There are the same number of paths in each group
2. The maximum exceedance is 2 = n, because this group contains all paths with n down steps below the horizon
3. There are n+1 groups because we are including the group with 0 exceedance

As for how many valid paths there are to 2n,
The total possible paths is given by: $\binom{2n}{n}$ because we know there will be 2n total steps and n up steps, so the path is totally determined by the placement of the up steps (n). Thus, it is a problem of 2n choose n.

However, we are only interested in the number of "valid" paths, aka the ones with zero exceedance (bc this group contains the paths that follow the rules for a Dyck path, aka they do not go below the horizon).

Thus, we divide by the total number of groups to get: $C_n = \frac{1}{n+1}\binom{2n}{n}$
Which is the combinatorial equation for the Catalan numbers!

Therefore, we have shown how we get the formula from an example to understand what all the terms mean.

Now we are ready to take on an even trickier problem. We are interested in the relationship between 321-avoiding permutations of S_n and Dyck paths of length 2n.

I will give an example that is easy to picture to help us consider this problem intuitively.

Suppose there are K penguins, k >= 3 and they are all different heights. We want to know how many ways there are to order the penguins in a line, left to right, so that we cannot find any three that are arranged tallest to shortest (this is like 321-avoiding).

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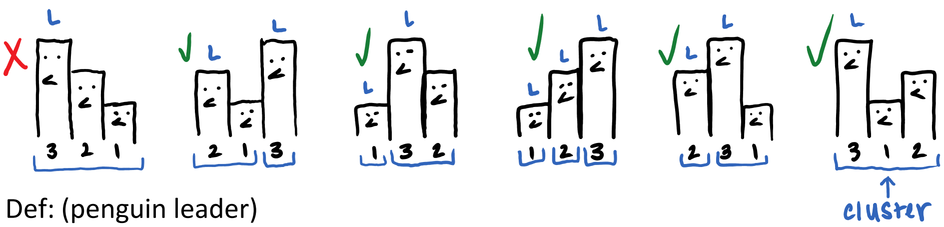


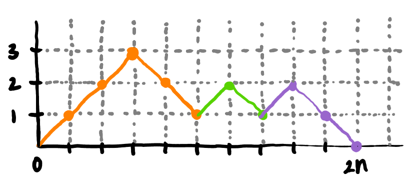
Def: (penguin leader)
- A penguin is a leader if it is taller than all penguins to its left
- → if a penguin is not a leader then it must be shorter than all penguins to its right

Key penguin property:
- The penguins will naturally huddle together for warmth leftwards towards the nearest leader; this arranges them into clusters.

Example: Let S = 31452.
The penguins will arrange themselves thus: (3 1) (4) (5 2)
Where clusters are enclosed in parentheses, and the leaders are the first member of each cluster.

We will now draw the Catalonian mountain range corresponding to S. The rules are:
1. Height of the peak = the relative height (in terms of shortness) of each leader with respect to <u>all</u> penguins to its <u>right</u>
2. Relative depth of the valley below the peak = size of the cluster

Example continued:
cluster size: $\overbrace{(3\ 1)}^{2}\overbrace{(4)}^{1}\overbrace{(5\ 2)}^{2}$



1. The first leader, 3, is the 3rd shortest of all 5 penguins, and it's in a cluster of 2 penguins
   -> mountain of height 3 and valley of length 2
2. The next leader, 4, is the 2nd shortest of the penguins to its right, and it's in a cluster of 1 penguin
   -> mountain of <u>overall</u> height 2 and valley of length 1
3. The final leader, 5, is also the 2nd shortest of the penguins to its right, and it's in a cluster of 2 penguins
   -> mountain of <u>overall</u> height 2 and valley of length 2

Thus, we have shown an example illustrating the bijection between 321-avoiding permutations of S_n and Dyck paths of length 2n.

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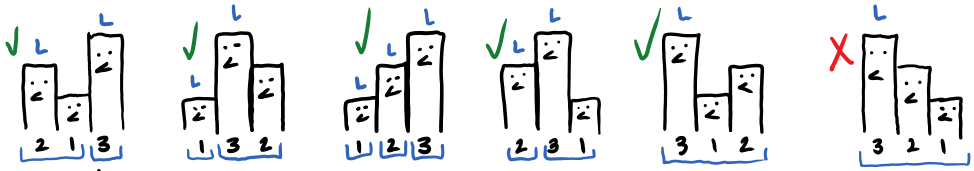
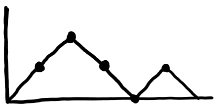
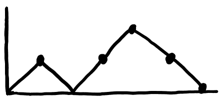
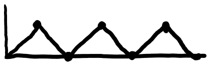
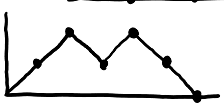
We can draw the Dyck paths corresponding to the penguin huddles I drew earlier:

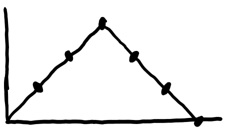


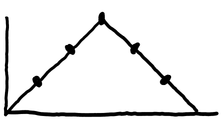
S = (21)(3) 

S = (1)(32) 

S = (1)(2)(3) 

S = (2)(31) 

S = (312) 

S = (321) 

What is interesting is that we get a unique Dyck path for each 321-avoiding arrangement, except for the 321 arrangement, further illustrating the bijective relationship between 321-avoiding permutations of Sn and Dyck paths.

Recall: consider an equivalence relation on the set R(w) of all reduced decompositions of a permutation w. We know that given any two reduced decompositions of minimal length a,b ∈ R(w), we can convert a to b by successive applications of the Coxeter relations:

C1: S_i S_j= S_j S_i, |i -j|≥2

C2: S_i S_(i+1) S_i= S_(i+1) S_i S_(i+1)

Def (fully commutative element):

- If a can be converted to b only using C1, then we call a and b C1-equivalent, or fully commutative

Theorem:

- Let w ∈ S_n. If w is 321-avoiding then any two reduced decompostions of w are fully commutative