q-analogs notes

We have met some q-analogs so far, but we haven't referred to them as such yet.

<u>Def</u>: A q-analog of a combinatorial object O is an object $O(q)$ such that $\lim_{q \to 1} O(q) = O$.

Q: What is O?

A: O can be a variety of things. It can be a number, a definition, or a theorem.

Here is an example of a common/important q-analog of $n \in \mathbb{N}$.

<u>ex</u>: $[n]_q = 1 + q + q^2 + \ldots + q^{n-1}$ $(n \in \mathbb{N})$

It is clear that $[n]_1 = n$.

Here is another example of a q-analog of n.

<u>ex</u>: $\dfrac{1-q^n}{1-q}$

note: In this example, we cannot just substitute $q=1$. Instead, we need to take the limit of the expression as q approaches 1.

Another important example of a q-analog is commonly referred to as the <u>q-factorial</u>.

<u>Def</u>: The q-factorial is represented by

$$[n]_q! = [1]_q[2]_q \dots [n]_q$$

We will now look at the generating function for inversions, which is Theorem 3.2.1 in the textbook.

<u>Thm</u>: For $n \geq 0$, we have

$$\sum_{\pi \in P[n]} q^{\mathrm{inv}\,\pi} = (1)(1+q)(1+q+q^2)\dots(1+q+q^2+\dots+q^{n-1})$$

We can prove that this generating function for inversions is a q-analog for $n!$

<u>Pf</u>: This proof can be conducted by inducting on n.

note: skipping the trivial base case

Every $\pi \in P([n])$ can be uniquely obtained from a $\sigma \in P([n-1])$. We can do this by inserting n into one of the n spaces between the elements of σ (which includes the space before σ_1 and the space after σ_{n-1}).

→

Let σ^i be the result of placing n in the i^{th} space from the right, where the space after σ_{n-1} is considered to be space 0. Then, we get

$$\operatorname{inv}\sigma^i = i + \operatorname{inv}\sigma$$

Now that we have this equation, we can use induction to get

$$\sum_{\pi \in P([n])} q^{\operatorname{inv}\pi} = \sum_{\sigma \in P([n-1])} \sum_{i=0}^{n-1} q^{\operatorname{inv}\sigma^i}$$

$$= \sum_{\sigma \in P([n-1])} q^{\operatorname{inv}\sigma} \cdot \sum_{i=0}^{n-1} q^i$$

$$= (1+q)(1+q+q^2)\ldots(1+q+q^2+\ldots+q^{n-1})$$

This is what we intended to prove. $\square$

Note: If we plug in $q=1$ to the result of the previous proof, we get

$$\#P([n]) = \sum_{\pi \in P([n])} 1 = n!$$

We have already discussed q-analogs involving permutations. However, they may also exist for combinations.

$\longrightarrow$

For integers $0 \leq k \leq n$, we can define the q-binomial coefficients or <u>Gaussian polynomials</u> as

<u>Def</u>: $$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

note: We can let this function be zero if we have $k<0$ or $k>n$.

<u>ex</u>: $$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{[4]!}{[2]!\,[2]!}$$

$$= \frac{[4]\,[3]}{[2]\,[1]}$$

$$= \frac{(1+q+q^2+q^3)(1+q+q^2)}{(1+q)}$$

$$= 1+q+2q^2+q^3+q^4$$

Now we will look at another theorem. This theorem gives two q-analogs for the ordinary binomial recursion. This demonstrates the general principle that q-analogs may not necessarily be unique.

$\longrightarrow$

This is theorem 3.2.3 in the textbook.

Thm: $\begin{bmatrix} 0 \\ k \end{bmatrix}_q = \delta_{0,k}$

for $n \geq 1$, we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

$$= \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

Now let's conduct a proof of this theorem.

Pf: note: the initial condition (base case) is trivial.
We can start by proving the first recursion for the q-binomial.
(We can prove the other one, but for the sake of time we'll just focus on the first).
If we use the definition in terms of q-factorials as well as finding the common denominator, we get

$$q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} = \frac{[n-1]!}{[k]!\,[n-k]!}\left(q^k[n-k]+[k]\right)$$

$$= \frac{[n-1]!}{[k]!\,[n-k]!\,[n]}$$

This is what we wanted. $\longrightarrow$ $= \begin{bmatrix} n \\ k \end{bmatrix}$

$\square$

We can also give a q-analog of the Binomial Theorem. Let q, t be variables.

This new theorem is Theorem 3.2.4 in the textbook.

Thm: For $n \geq 0$, we have that

$$(1+t)(1+qt)\dots(1+q^{n-1}t) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q t^k$$

Now let's prove this theorem.

Pf: We can prove this using induction on n. The base case $n=0$ is simple to check. For $n>0$, we can use the second recursion from the previous result in conjunction with the induction hypothesis.

$$\begin{aligned}
\sum_k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q t^k &= \sum_k q^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q t^k + \sum_k q^{\binom{k}{2}+n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q t^k \\
&= (1+t)(1+qt)\dots(1+q^{n-2}t) + q^{n-1}t \sum_k q^{\binom{k-1}{2}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q t^k \\
&= (1+t)(1+qt)\dots(1+q^{n-2}t) + q^{n-1}t(1+t)(1+qt)\dots(1+q^{n-2}t) \\
&= (1+t)(1+qt)\dots(1+q^{n-1}t)
\end{aligned}$$

$\uparrow$

This is what we wanted. $\square$

Now, we will talk about <u>lattice paths</u>.

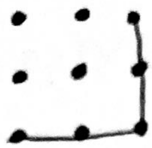
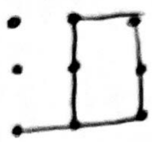
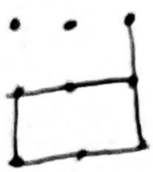
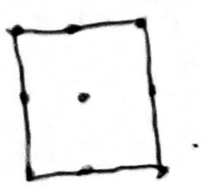
Let's consider lattice paths from the origin to the point (m,n). Here, m and n are non-negative integers and the allowable steps are east and north.

The number of paths can be represented by $\binom{m+n}{m}$, because we have to take $m+n$ steps in total, of which m must go east and n must go north.

For each path p, there exists a specific area $A(p)$ enclosed between the path, the x-axis, and the line represented by $x=m$.

<u>ex</u>: The following figures show the six paths for $m=n=2$ and the area that is enclosed in each case.

→

Path	Area
	0
	1
	2
	2
	3
	4

These are the example lattice paths.

We can take a generating function in the variable q for the areas of these lattice paths. A path with area A contributes q^A to the sum, and hence we get

$$1+q+2q^2+q^3+q^4 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q$$

Furthermore, there is a theorem (6.9 in textbook) about these lattice paths.

Thm: Let $\mathcal{P}$ be the set of lattice paths from $(0,0)$ to (m,n) using only northerly and easterly steps. For $p \in \mathcal{P}$, let $A(p)$ be the area which is enclosed by p, the x-axis, and line $x=m$.

$$\sum_{p\in\mathcal{P}} q^{A(p)} = \begin{bmatrix} m+n \\ m \end{bmatrix}_q$$

Let's prove this theorem.

$\longrightarrow$

Pf: Let's denote the left-hand side as $F(m,n)$.
It is clear that $F(0,n) = F(m,0) = 0$.
Now, let's consider $F(m,n)$.
We have two cases:

1) If the last step on the path p is headed north, then it's a path from $(0,0)$ to $(m, n-1)$ followed by a north-heading step, and the last step does not change the area.

2) If the last step is headed east, then it's a path p from $(0,0)$ to $(m-1, n)$, followed by an east-heading step, which adds n to the area.

Therefore, $F(m,n) = F(m, n-1) + q^n F(m-1, n)$.

Now, simple induction on
$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1$, $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$ for $0 < k < n$
gives us the result we're looking for.

$\square$