Chapter 1.1.1 - Definition of Coxeter System

Def (Coxeter System (W,S)).

A coxeter system is just a group w/ special presentation

let $S = \{s_1, \dots, s_n\}$ then $W = \langle S \mid s_i^2 = e \ \forall i, \ (s_i s_j)^{m_{ij}} = e \rangle$ where $m_{ij} \geq 0$ · S_n = coxeter group · each of i^2 is identity

W is equivalent to $\underbrace{s_i s_j s_i}_{m_{ij}} = \underbrace{s_j s_i s_j}_{m_{ij} \text{ elements}}$

MORE IN DEPTH:

Now we will look more in depth at identity element

- relations for $s = t$ will be written in form $s^2 = id$
  - quadratic relations: relation $(st)^{m_{st}} = id$ for $s \neq t \in S$ is equivalent (under the quadratic relations) to the braid relation $(\underbrace{sts\dots}_{m_{st}} = \underbrace{tst\dots}_{m_{st}})$

More definitions which may make above clearer

m_{st} = order of element st → when $m_{st} = \infty$ NO corresponding relation between s and t

elements of S = simple reflections • elements of W = conjugate to S, reflections

REMARK: group W when there exists a coxeter system (W,S) is a coxeter group

coxeter group can be equipped with structure of Cox. group in many ways:

1. any conjugate of S can be used as set of simple reflections
   ↳ is possible for same grp to be described as cox. group using 2 cox systems w/ diff. ranks → learn more later
2. for each $w \in W$ can write $w = s_1 \dots s_k$ for some $s_1, \dots, s_k \in S$ ← an expression for w of length k

when $w = s_1 \dots s_k$ can write seq. as $\underline{w}$

↳ $\underline{w}$ indicates elmt $w \in W$ and choice of expression for W

Coxeter graph of Coxeter System (W,S):

CHAPTER 1.1.2 - Type A

Definition: Coxeter system of type A_{n-1}, $n \geq 2$ is given by the cox. graph: s_1 — s_2 — s_3 ...... s_{n-2} — s_{n-1} (0 edges btws s_1 and s_3 so $m_{st} = 2$)

generates the set $S = \{s_1, s_2, \dots, s_{n-1}\}$ and relations

* $s_i^2 = id$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, $s_i s_j = s_j s_i$ if $|i-j| > 1$

S_n = group of permutations of $\{1, 2, \dots, n\}$, symmetric group

s_i = generator, corresponds to the adjacent transposition $(i, i+1)$

* (the coxeter group) is isomorphic to S_n

Drawing coxeter graph: (graph is compact way to say S_n

S → • vertices correspond generators → should be able to look at graph and know what m_{st} it is

• edges correspond to "m_{st}"

RULES: edge must directly touch each vertex

1) If $m_{st} = 2$ then no edge btw vertex s and vertex t
2) If $m_{st} = 3$ then 1 edge btw "
3) If $m_{st} > 3$ then vertex s and t are joined by a labeled edge

s_1 to s_3 = $m_{st} = 2$ $m_{st} = 3$ for s_2 and s_3

* $s_i^2 = id$ $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, $s_i s_j = s_j s_i$ if $|i-j| > 1$

· this graph is a compact way to say S_n
↳ S_n = group of permutations of $\{1, 2, 3, \ldots, n\}$
· s_i = generator, corresponds to the adjacent transposition $(i, i+1)$.

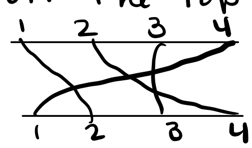
★ Coxeter group is isomorphic to S_n

EXERCISE 1.5

objective: confirm that the set of all reflections agrees with the set of transpositions (i,j). For each transposition (i,j) with $i<j$, find an expression (i,j) with length $2(j-i)-1$

Common way to describe a permutation $w \in S_n$

strand diagram notation (most important for us):
permutation $w \in S_n$ depicted as a diagram with n strands where each strand connects from i on the bottom line to $w(i)$ on the top line. Again looking at $w = 4132$:

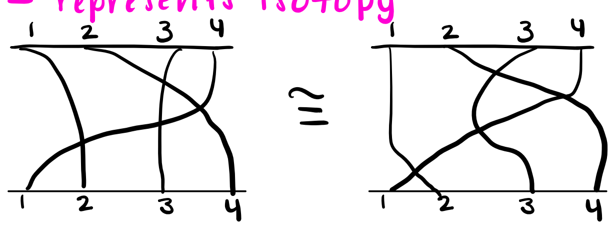


bottom = i
so $i=1$ and $w(1)=4$ → can see how 1 connects to 4

Note: stacking diagrams vertically multiplies permutations
↳ xy is x on top of y

③ Relates most closely to the coxeter presentation of S_n
③ does not represent $w \in S_n$ but it is an expression $\underline{w}$ for a permutation
∘ can build any suitably generic strand diagram by vertically stacking crossings (diagrams with a single-crossing that corresponds to the simple reflections $(i, i+1)$)
∴ length of the expression = # of crossings

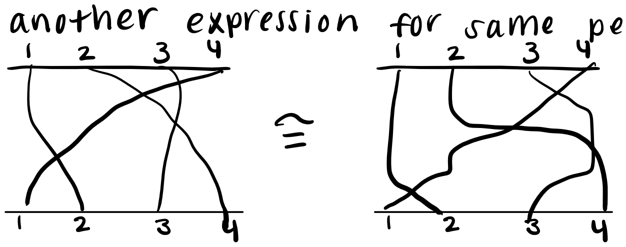
More strand diagram examples:
≅ represents isotopy



corresponds to the expression above

showing that it only matters where you start and end up

another expression for same permutation:



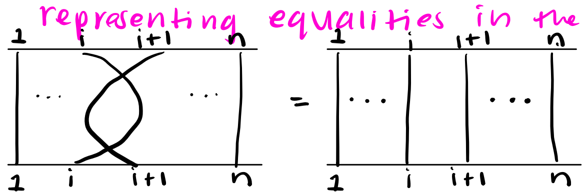
corresponds to $\underline{w} = (s_3, s_2, s_3, s_1)$

→ only say this too if I have time

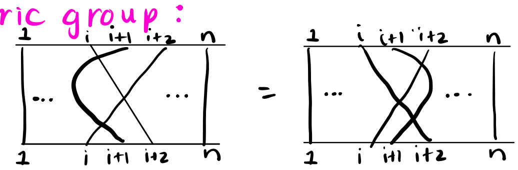
Exercise 1.6: previous example found expression for transposition (i,j) of length $2(j-i)-1$ (when $i<j$). Now draw this expression as a strand diagram

Smthg to think abt: → think about what other expressions for (i,j) of this length you can find → can you find any shorter expressions for (i,j)?

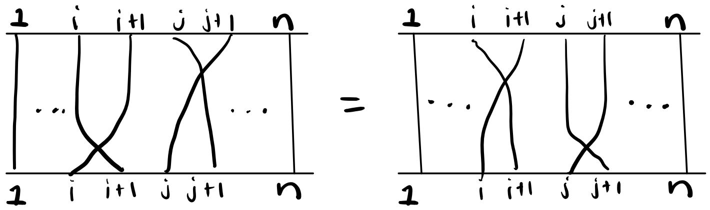
Coxeter relations become following manipulations of strand diagrams representing equalities in the symmetric group:



} $s_i^2 = e$



(braid relation)



} $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

These graphs visually represent:

- $s_i = (i, i+1) =$ swap i and $i+1$
- $S_n = \langle s_1, \dots, s_{n-1} \mid \cdot\, s_i^2 = e,\ \cdot\, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ \cdot\, s_i s_j = s_j s_i \ \ |i-j|>1 \rangle$

ex.) $|s_i s_{i+1}|^3 = e$

if you expand the equation as $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

REMARK: a strand diagram is *generic* if:

- it has only transverse intersections with no triple intersections
- no 2 crossings occur at the same height

We can obtain an expression from a generic strand diagram by reading off the crossing in order of their height

↳ thus an isotopy of diagrams which does not change the order of the crossings will not change the resulting expression

*typically think of 2 diagrams that are isotopic in this way as being the same

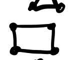
- An isotopy which does not create triple intersections but does change order of crossings will produce a diff. expression related to the original by applications of the braid theorem

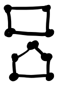
EXAMPLE 1.2.7

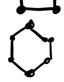
QUESTION: *Dihedral Groups*. Let L_1 and L_2 be straight lines through the origin $\mathbb{E}^2$ (Euclidean plane). Assume that the angle btw them is π/m for some integer $m \geq 2$. Let r_1 be the orthogonal reflection through L_1, and similarly for r_2. Then $r_1 r_2$ is a rotation of the plane through the angle $\frac{2\pi}{m}$ and, hence, $(r_1 r_2)^m = e$

D_{2n} = Dihedral groups symmetries of regular n-gon $2n$ elements

ex: D_6 = 

D_8 = 

D_{10} = 

D_{12} = 

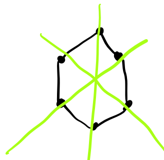
$a^n = 1$ (order n) $b^2 = 1$

$bab^{-1} = a^{-1}$

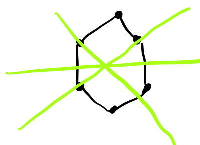
element a = rotation

↓

take D_{10}: 

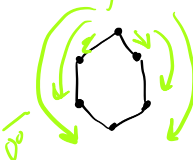


3 reflection ①
(2 conjugacy classes)



+ 3 reflections ②

→ relections can go through 2 vertices or 2 edges



identity rotations → does nothing
+2
+2
+1 order 2 rotations

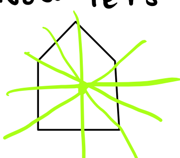
③ 3 involutions

↳ rotation around 180° is in the center so the center has order 2

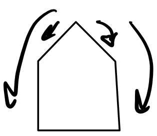
Def (involutions) = element of order 2

* Something similar happens D_{2n} when n is even
  - 3 classes of involution
    - ↳ one of which is an order 2 in the center
    - ↳ get the identity element in a function of pairs of rotations

Now lets look at D_{2n} when n is ODD



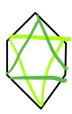
5 relflections (order 2)
1 conjugancy class → goes through vertex and edge on the other side



identity rotation of order 1
rotations a fifth of a revolution of order 5 (just say this)
2 order 5
2 order 5
center is TRIVIAL (only contains 1)
↳ no rotation of 180° so no center

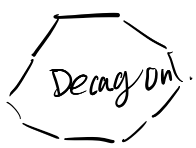
- 1 conjugacy class for involutions

Another interesting property for even dihedral groups:



$D_6 \subseteq D_{12}$ (D_6 is contained in D_{12} and is a subgroup)
$\{1, -1\} \leq D_{12}$ → order 2
* $D_{12} = D_6 \times \{1, -1\}$

Note: whenever we have group D_{4n} n odd ($D_4, D_{12}, D_{20}\ldots$) all split as a product of 2 groups

ex: $D_{20} =$ 

→ 2 pentagons (cannot draw this

On the syllabus: diagrams of symmetries of dodecahedron and hyperbolic planes and affine planes

Note: any dihedral group is generated by 2 involutions

Example 1.2.8: symmetry groups of regular polytopes

Symmetries of a regular m-gon in the plane = m rotations and the m orthogonal reflections through a line of symmetry

As we said in example 1.2.7: the symmetry group = dihedral group $I_2(m$

Def: 2-dimensional regular polygons have their counterparts in higher dimensions among the regular polytope

* d-dimensional convex polytope is regular if given 2 nested seq of faces $F_0 \subseteq F_1 \subseteq \cdots F_{d-1}$ (dim $F_i = i$) there is some isometry of d-space that maps the polytope onto itself and maps the first seq. of faces to the other one

a more concrete way to think of this is: ↳ Note: symmetry groups of regular polytopes are always Coxeter groups

example: In dimension 3, a polytope is regular if all the faces are the same and around each vertex, the way the faces meet are the same

Full classifications with corresponding Coxeter groups as symmetry groups:

dimension	regular polytope	Coxeter group
d	simplex	A_d
d	cube	B_d
d	hyperoctahedron	B_d
2	m-gon	$I_2(m)$
3	dodecahedron	H_3
3	icosahedron	H_3
4	24-cell	F_4
4	120-cell	H_4
4	600 cell	H_4

maybe don't write it all out for the sake of time

say: to picture the link btw polytope and group:

take the dodecahedron:

↳ 15 planes of symmetry which subdivide its boundary into 120 congruent triangles

↳ the orthogonal reflections through the planes generate the full symmetry group W and this group acts simply transitively on the triangles

carrying on with this example... thinking in a geometric perspective

what are the coxeter generators?

· fix any of the 120 triangles and call it the fundamental region

take as S the 3 reflections in the "walls" of this triangle

then (W, S) is a coxeter system

↳ the dihedral angles in the corners of any Δ are $\pi/2$, $\pi/3$, and $\pi/5$ where denominators are defining numbers $m(s,s')$ of coxeter system of type H_3

CONTINUE WITH LECTURE IF I HAVE TIME

1.1.3 - Type B

Definition : Coxeter system of type B_n : $n \geq 2$ is given by coxeter graph:

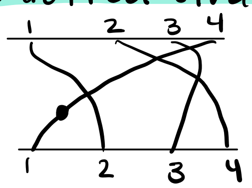
Coxeter group is isomorphic to the signed symmetry group (denoted by SS_n) → the group of permutations w of $\{\pm 1, \pm 2, \ldots, \pm n\}$ s.t. $w(-i) = -w(i)$

↳ isomorphism sends t to the transposition $(-1, 1)$ and s_i to the product of transpositions $(i, i+1)(-i, -(i+1))$

Exercise 1.12: Verify that $s_1 t s_1 t = t s_1 t s_1$ in SS_n. Verify that the coxeter group of type B_n is isomorphic to SS_n for $n=2$ and $n=3$ ← Q

A: Since SS_n is a subgroup of S_{2n} we can use one-line, cycle, or strand notation to describe elements of SS_n but there are 2 more reasonable diagrammatic approaches (just say that)

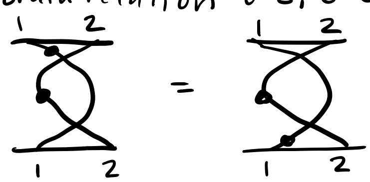
① **dotted strand diagram:** dot corresponds to sign flip



→ permutation w sending $1 \mapsto -4$, $2 \mapsto 1$, $3 \mapsto 3$, $4 \mapsto 2$ ∴ it also sends $-1 \mapsto 4$, $-2 \mapsto -1$ and so on

in terms of simple reflection: $w = s_2 s_2 s_3 s_1 t$

Braid relation $t s_1 t s_1 = s_1 t s_1 t$



∗ Dotted strand diagram (DSD)
→ this note is just for me

Exercise 1.13
write down diagrammatic relations for dotted strand diagrams akin to the relations in (FIGURE A)

Note: say → { advantage of dotted strand diagram: easy to draw and multiply!
cons " " " " : obscure coxeter structure

write ~ •Simple reflections correspond to elementary dotted strand lines with s_i to a crossing and t to a dot on the first strand
↳ thus a DSD corresponds to an expression ONLY when all dots appear on the first strand
above but true for ^ ex. where this is not true:

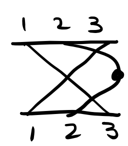
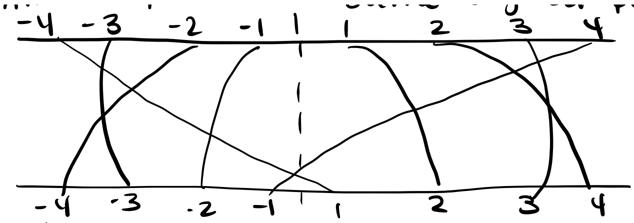


FIGURE B

Exercise 1.14:
let x be signed permutation in figure B (just point on the board)
Find an expression for x (find a dotted strand diag. for x where dots only appear on the first strand). Compare x with $s_2 x s_2$:
- which has a DSD w/ fewer crossings?
- which one has a shorter expression?

② **Mirror strand diagrams / mirror pictures** (connects to ① above)
·special kind of 2n-strand diagram.
·each strand connects i to $w(i)$ but the picture is required to be symmetric across a central axis (the mirror)
mirror picture of same signed permutation $w = s_3 s_2 s_1 s_3 t$:



this best reflects the coxeter structure

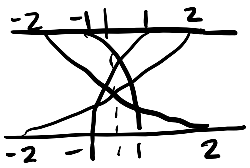
each mirror pair of crossings corresponds to a simple reflection s_i

↳ each crossing which occurs on top of the mirror (dashed line) corresp. to simple reflection t

↳ length of expression can be computed by counting the # of crossings on or to the right of the mirror

↳ disadvantage: hard to draw by hand

$ts_2ts_1 = s_1ts_2t$ in braid notation:



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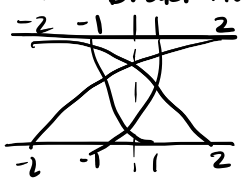

Remark: Mirror strand diagrams typically not generic in relation to def. of generic in terms of strand diagrams as defined above

EXAMPLE 1.29 Reflection Groups:

- dodecahedron we discussed earlier shows how a certain finite coxeter group can be realized as a group of geometric transformations generated by reflections
  - ↳ true of all finite coxeter groups

Say: 2 most important classes of infinite coxeter groups are defined in terms of their realizations as reflection groups

↳ affine and hyperbolic coxeter groups (will discuss in a diff. lecture)

ex 1.2.10: Weyl groups of root systems

For $\alpha \in \mathbb{E}^d \setminus \{0\}$, let σ_α denote orthogonal reflection in the hyperplane orthogonal to α. In particular, $\sigma_\alpha(\alpha) = -\alpha$

Definition: finite set $\Phi \subset \mathbb{E}^d \setminus \{0\}$ is a crystallographic root system if it spans $\mathbb{E}^d$ and for all $\alpha, \beta \in \Phi$: (these 3 hold)

1. $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$
2. $\sigma_\alpha(\Phi) = \Phi$
3. $\sigma_\alpha(\beta)$ is obtained from β by adding an integral mult. of α

Group W generated by reflections $\sigma_\alpha, \alpha \in \Phi$ is Weyl Group of Φ

Definition: A pair B, N of subgroups of a group G is called a BN-pair (or Tits system) if these 4 axioms hold:

1. $B \cup N$ generates G, and $B \cap N$ is normal in N
2. $W \overset{\text{def}}{=} N/(B \cap N)$ is generated by some set S of involutions
3. $s \in S, w \in W \rightarrow BsB \cdot BwB \subseteq BswB \cup BwB$
4. $s \in S \rightarrow BsB \cdot BsB \neq B$

↳ Axioms show us the set S is uniquely determined and (W,S) is a Coxeter system

↳ W = Weyl Group, $\#|S|$ = rank of BN-pair

Bruhat decomposition: $G = \biguplus_{w \in W} BwB$

Thus: Weyl group W acts as indexing set for the partition of the group G into pairwise disjoint subsets

↳ double cosets w.r.t. the subgroup B

say: we will see how the partition induces a partial order on the set W
later on in the course