OLIVIA SOLANOT

PRINCIPLE OF INCLUSION & EXCLUSION

MOTIVATION

1. Considered as a basic counting tool
2. Applications: counting derangements, counting number of onto functions, counting intersections, & Euler's function (just to name a few)

(INTRODUCTORY) EXAMPLE

(Let us consider a class of students)

Group A - 20 students study Algebra

Group B - 25 students study Probability

→ (we know that) 8 students study both

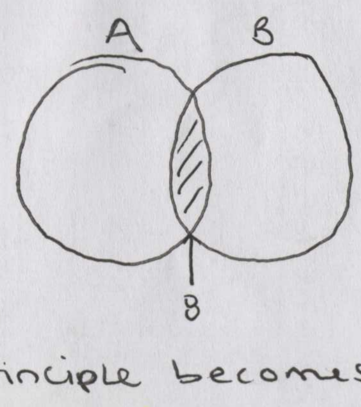
(If we are asked how many students study Algebra or Probability, we may tend to simply add the two)

$20 + 25 = 45$

(this does not take into account students who have studied both.)

$20 + 25 - 8 = 37$

(this can be represented w/ a venn diagram)



The union of two sets can be represented as:

$$|A \cup B| = \underbrace{|A|}_{20} + \underbrace{|B|}_{25} - \underbrace{|A \cap B|}_{8}$$

(This principle becomes more avid in case of 3 sets)

3 sets case:

$$|A \cup B \cup C| = |A| + |B| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

(NOW we will go into the generalization for n sets)

Generalization for n sets

(for context) Binomial Theorem:

(tells us how to expand expressions of the form) $(a+b)^n$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Theorem 2.1 In its general form, the principle of inclusion and exclusion states that for finite n sets $A_1, \ldots, A_n$ the following holds.

$$\left| \bigcup_{1 \le i \le n} A_i \right| = \sum_{1 \le i_1 \le n} |A_{i_1}| - \sum_{1 \le i_1 \le i_2 \le n} |A_{i_1} \cap A_{i_2}|$$

$$+ \sum_{1 \le i_1 \le i_2 \le i_3 \le n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \ldots + (-1)^{n+1} \left| \bigcap_{i=1}^{n} A_i \right|$$

PROOF (THEOREM 2.1)

suppose $x \notin A_1 \cup A_2 \ldots \cup A_n$

therefore contribution of x to $A_1 \cup A_2 \ldots \cup A_n$ is 0 in L.H.S.

In R.H.S. it is also 0 as x is not present in any of the sets.

count becomes 0

Let $x \in A_1 \cup A_2 \ldots \cup A_n$

→ L.H.S. count = 1 (because any element will be present only once in a set)

→ R.H.S. x will either be present in each individual set or not.

assume x belongs to k such sets from $A_1, \ldots, A_n$

we have

$$\sum_{1 \le i_1 \le n} |A_{i_1}| = k$$ (as x is present in exactly k sets)

$$\sum_{1 \le i_1 \le i_2 \le n} |A_{i_1} \cap A_{i_2}| = \binom{k}{2}$$ (as we choose any 2 pairs out of k in $\binom{k}{2}$ ways)

$$\sum_{1 \le i_1 \le i_2 < i_3 \le n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| = \binom{k}{3}$$ (and so on)

R.H.S. becomes

$$\binom{k}{1} - \binom{k}{2} + \binom{k}{3} - \ldots + (-1^{k-1})\binom{k}{k}$$

$$= \binom{k}{0} - \binom{k}{0} + \binom{k}{1} - \binom{k}{2} + \binom{k}{3} - \ldots + (-1^{k-1})\binom{k}{k}$$

$$= \binom{k}{0} - \left[\binom{k}{0} - \binom{k}{1} + \binom{k}{2} - \binom{k}{3} + \ldots + (-1^{k})\binom{k}{k}\right.$$

$$= 1 - [1 + (-1)]^k \implies 1$$ completes the proof

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APPLICATIONS

COUNTING PROBLEMS

How many positive integers less than 100 is not a factor of 2, 3, and 5?

(for solving this at first we have to find the # of positive integers less than 100 which are divisible by 2 or 3 or 5.)

Let A = The set of elements which are divisible by 2

Let B = The set of elements which are divisible by 3

Let C = The set of elements which are divisibly by 5

(The principle of inclusion & exclusion gives us)

$$|A \cup B \cup C| = |A| + |B| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

$$= [99/2] + [99/3] + [99/5] - [99/2*3] - [99/3*5] - [99/2*5]$$

$$+ [99/2*3*5]$$

$$= 49 + 33 + 19 - 16 - 6 - 9 + 3 = 73$$

total # of integers which are not a factor of 2 or 3 or 5 is $99 - 73 = \underline{\underline{26}}$

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DERANGEMENTS

... a derangement is a permutation of 1, 2, 3, ... n such that none of the elements appear in their original position.

ex 123 → 312 & 231

(we can use the principle of incl & excl to count # of derangements among all possible permutations)

- total # of permutations = $n!$
- number of ways in which any one is at correct position = $\binom{n}{1}(n-1)!$
- (similarly for the number of ways any two are at correct position) = $\binom{n}{2}(n-2)!$

total # of derangements:

$$= n! - \left[\binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! \ldots + (-1^n)\binom{n}{n}(n-n)!\right]$$

$$= n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1^n)\frac{1}{n!}\right]$$

DERANGEMENTS EXAMPLE

(A sequence of n theatergoers want to pick up their hats on the way out. However, the deranged attendant does not know which hat belongs to whom and hands them out in a random order.
→ what is the probability that no one gets their own hat?)

Formally:

random permutation $\pi : [n] \to [n]$

asking the prob. that $\forall i ; \pi(i) \neq i$

Theorem 1 The probability that a random permutation $\pi : [n] \to [n]$ is a derangement is $\sum_{k=0}^{n} (-1)^k / k!$, which tends to $\frac{1}{e} = 0.3678\ldots$ as $n \to \infty$

PROOF

X = set of all $n!$ permutations

A_i = set of permutations that fix element

$$A_i = \{ \pi \in X \mid \pi(i) = i \}$$

in simple counting : $(n-1)!$ perms. in A_i
(since by fixing i, we still have $n-1$ elements to permute)

$\bigcap_{i \in I} A_i$ = perms. where all elements of I are fixed
→ # of perms is $(n - |I|)!$

By Inclusion - Exclusion we know that the # of perms with some fixed point is

$$\left|\bigcup_{i\in I} A_i\right| = \sum_{\emptyset = I \subseteq [n]} (-1)^{|I|+1} \left|\bigcap_{i\in I} A_i\right|$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)!$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \frac{n!}{k!}$$

$$\sum_{k=1}^{n} (-1)^{k+1} / k!$$

(by taking the complement)

probability that there is no fixed point is:

$$1 - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} = \sum_{k=0}^{n} \frac{(-1)^{k}}{k!}$$

[redacted]

Euler's ϕ function

is an arithmetic function that counts the total number of positive integer which are relatively prime to n and are less than n.

→ a.k.a. Totient Function

(context)

relatively prime: x is rel. prime ~~to~~ to N if GCD(N, x) = 1

if N is prime the $\phi(N) = N-1$

EXAMPLE

Let N be an integer such that $N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_r^{a_r}$

where $p_1, p_2, p_3, \ldots, p_r$ are all prime #'s

Find the value of $\phi(N)$

SOLUTION

Let S_i = set of integers less than N which are divisible by p_i $(1 \le i \le r)$

$|S_i| = N/p_i$

(the # of integers which are factors of atleast one prime)

↳ $$S = \sum_{1 \le i \le r} \frac{N}{p_i} - \sum_{1 \le i,j \le r} \frac{N}{p_i p_j} + \sum_{1 \le i,j,k \le r} \frac{N}{p_i p_j p_k} - \cdots$$

(by definition) $\phi(N) = N - S$

$$= N - \sum_{1 \le i \le r} \frac{N}{p_i} + \sum_{1 \le i,j \le r} \frac{N}{p_i p_j} - \sum_{1 \le i,j,k \le r} \frac{N}{p_i p_j p_k} + \cdots$$

$$= N \prod_{i=1}^{r} \left[1 - \frac{1}{p_i}\right]$$

EULER EXAMPLE

$N = 100 = (2^2)(5^2)$

$\phi(100) = 100(1/2)(4/5) = 40$

whereas to compute $\phi(100)$ manually, you would need to compute and compare the prime factorizations of all #'s below 100 with 100 and record which ones are relatively prime

APPLICATION: # of SURJECTIONS

m teachers

n children $m \geq n$

(each teacher gives one random child a cookie)

(what is the prob. that all n children get at least one cookie?)

Theorem 2: The probability that all n children get cookies is $\sum_{k=0}^{n-1} (-1)^k \binom{n}{k} \left(1 - \frac{k}{n}\right)^m$

PROOF

function $f: [m] \to [n]$ is a surjection if it covers all element of $[n]$

n^m total functions

$A_i = \{f : [m] \to [n] \mid \forall j : f(j) \neq i\}$

(this represents the set of functs that leave element i unconvered)

number of functions is $(n-1)^m$ since there are $n-1$ choices for each $f(1), f(2), \ldots f(m)$

$\left|\bigcap_{i \in I} A_i\right| = (n - |I|)^m$

where $|I| \to$ forbidden choices for each function value

# of functions not surjections:

$$\left|\bigcup_{i=1}^{m} A_i\right| = \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|+1} \binom{n}{|I|} (n - |I|)^m$$

complement gives # of surjections:

$$n^m - \left|\bigcup_{i=1}^{m} A_i\right| = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} (n-k)^m$$

(dividing by n^m gives the desired probability)