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Eulerian Polynomials

Notation 0.1

a. S_n = set of all permutations of $[n]$
b. Values of $w \in S_n$ are denoted as w_i or $w(i)$
c. In one line notation this is represented as $w = w_1 w_2 w_3 \cdots w_n$
d. In two line notation this is represented as

$$w = \begin{pmatrix} 1 & 2 & \cdots & n \\ w_1 & w_2 & \cdots & w_n \end{pmatrix}.$$

Definition 1.1 Descents

Eulerian numbers are defined using descents, so let us define descents first.

$\boxed{\text{For } w \in S_n,\ i \in [n-1] \text{ is a } \textbf{descent} \text{ of } w \text{ if } w_i > w_{i+1}}$

Example 1.2
Permutation: 6 1 2 7 5 3 9 4 8 → 6 · 1 2 7 · 5 3 9 · 4 8
Because $6 > 1,\ 7 > 5,\ 9 > 4$

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 1 & 2 & 7 & 5 & 3 & 9 & 4 & 8 \end{pmatrix}$$

Descent $D(w) = \{i \in [n-1] \mid w_i > w_{i+1}\}$
Descent set $des(w) = \#D(w)$

So from the example $D(w) = \{1, 4, 5, 7\}$ and $des(w) = 4$

Definition 2.1 Eulerian numbers

In combinatorics, the Eulerian number $A(n, k)$ is the number of permutations of the numbers 1 to n in which exactly k elements are greater than the previous element.

$\boxed{A(n, k) = \#\{w \in S_n \mid des(w) = k\} \text{ for } 0 \le k \le n - 1}$

Example 2.2
$n = 3,\ k = 1$
$A(3, 1)$

We are looking for permutations with $k = 1$ descents
132, 213, 312, 231
In all of the above there is exactly 1 descent from $3 \to 2,\ 2 \to 1,\ 3 \to 1,\ and\ 3 \to 1$

Hence, $A(3, 1) = 4$ in this example
NB Change k = 1 2 3 4 5 6 7 -> 0 1 2 3 4 5 6

$n \setminus k$	~~1~~ 0	~~2~~ 1	~~3~~ 2	~~4~~ 3	~~5~~ 4	~~6~~ 5	~~7~~ 6
1	1						
2	1	1					
3	1	4	1				
4	1	11	11	1			
5	1	26	66	26	1		
6	1	57	302	302	57	1	
7	1	120	1191	2416	1191	120	1

$$A(n, k) = \#\{\text{permutations of } [n] \text{ with } k - 1 \text{ descents}\}$$

Example 2.3
Notice that $A(n, 0) = \#\{permutations\ with\ 0\ descents\} = 1$, which is exactly 1: $w = 1\,2\,3\,4 \cdots n$

Similarly $A(n, n - 1) = 1$ because there is only one permutation where the number of descents is equal to the number of elements: $w = n,\ n - 1,\ n - 2, \cdots,\ 1$

Proposition 2.4

$A(n, k) = (k + 1) \cdot A(n - 1, k) + (n - k) A(n - 1, k - 1)$

Example 2.5
$n = 5,\ k = 1,\ A(5, 1) = 26 = (1 + 1) \cdot A(4, 1) + (5 - 1) A(4, 0) = 2 \cdot 11 + 4 \cdot 1 = 22 + 4 = 26$

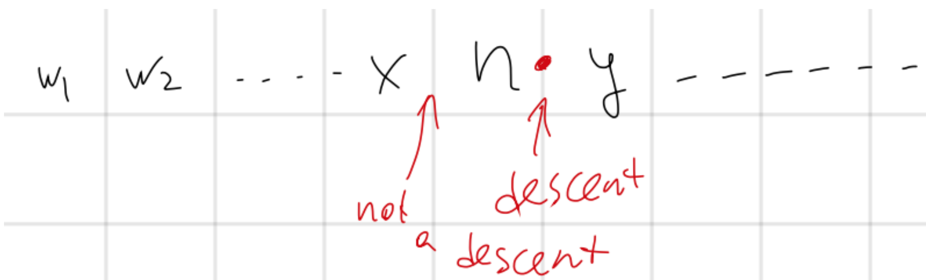
Proof of 2.4

Let $w \in S_n$

$des(w)$ either stays the same or drops by 1

- Case 1: descent stays the same when n is in the middle of a descent where $x > y$, so removing n preserves the descent between n and y so coefficient = k

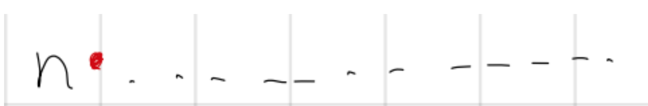
$des(\pi') = k$



Or n is the last letter, so the number of descents does not change so coefficient = k + 1

Hence number of options $= (k+1) \cdot A(n-1, k)$

- Case 2: the number of descents drops by 1. So here $x < y$



- Or n is the first letter

Case 2: $\mathrm{des}(\pi') = k-1$

Similar to above but we now can only put n in between places xy where $x < y$. The total number of these places is $(n-2)-(k-1)) = n-k-1$ (As there are $n-2$ places for descents in S_{n-1} and we are given that π' has $k-1$ descents). Like before we have an additional case

$$\pi' = (\cdots) \mapsto \pi = n(\cdots)$$

bringing the total places to put n into to $n-k$. □

Example

Where can we insert 6 into the following permutation while preserving the number of descents?
1 3 4 2 5

Answer:

we can remove 5 at the end which will preserve the number of descents

Ex. ^ 1 ^ 3 ^ 4 • 2 ^ 5

insert 6 ~~preserving~~ ~~#~~ descents: 4 places
to give 2 descents

Let n = 6, k = 2, n-k = 4

Generating function for $A(n, k)$

Problem that it is a triangle of numbers rather than a sequence, i.e. it depends on two parameters.

Def $A_n(x) = \sum_{k=1}^{n} x^k \cdot \underbrace{A(n, k)}$, $n \geq 1$
Eulerian numbers are the coefficients

$A_o(x) = 1$

In other words, $A_n(x) = \sum_{w \in S_n} x^{1+des(w)}$

Proof $\quad A_0(x) = \sum_{w \in S_n} x^{1+des(w)} = x^0 = 1$?

$$A_n(x) = \sum_{k=1}^{n} A(n,k) x^k = \sum_{k=1}^{n} \Big(\sum_{w \in A(n,k)} 1\Big) x^k \overset{(1)}{=} \sum_{w \in S_n} x$$

coeff in front of x^k is # of el w/ k-1 descents

$A(n,k)$ $\quad \coprod_{k=1}^{n} A(n,k) \overset{(1)}{=} S_n$

$des(w) =$

$\sum_{w \in S_n} x^{des(w)+1}$ $\quad$ coeff in front of x^{k+1} is # of el w/ k descents

maybe just start proof

$$A_n(x) = \sum_{k=1}^{n} A(n,k)\, x^k = \sum_{k=1}^{n} \underbrace{\Big(\sum_{w \in A(n,k)} 1\Big)}_{= des(w)} x^k = \sum_{w \in S_n} x$$

$= des(w)$ → number of elements with k-1 descents

disjoint union → $\coprod_{k=1}^{n} A(n,k) = S_n$

$\sum_{w \in S_n} x^{des(w)+1}$ → number of elements with k descends

This leads to an interesting proposition

Proposition 1.8. *For each $n \geq 0$,* Not going to prove fully just start

$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{m\geq 0} (m+1)^n t^m$$

Proof. I personally would leave this as an exercise with the hint to use the recursion given above. □

$$\sum_{m\geq 0} m^n \cdot x^m = \frac{A_n(x)}{(1-x)^{n+1}}$$

$A_0(t) = 1$
$A_1(t) = 1$
$A_2(t) = 1 + t$
$A_3(t) = 1 + 4t + t^2$
$A_4(t) = 1 + 11t + 11t^2 + t^3$
$A_5(t) = 1 + 26t + 66t^2 + 26t + t^4$
$A_6(t) = 1 + 57t + 302t^2 + 302t^3 + 57t^4 + t^5$

Proof by induction for each $n \geq 0$

Base case $n = 0$

$$\sum_{m>0} x^m = \frac{1}{1-x} \quad \text{— ok}$$

induction step: suppose the statement holds

$$\sum_{m\geq 0} m^n x^m = \frac{A_n(x)}{(1-x)^{n+1}}$$

Take derivative & multiply by x

$$\sum_{m\geq 1} m^{n+1} x^m = x\,\frac{A_n'(x)\cdot(1-x)^{n+1} + A_n(x)(n+1)(1-x)^n}{(1-x)^{2n+2}}$$

$$= \frac{x(1-x)A_n'(x) + (n+1)xA_n(x)}{(1-x)^{n+2}}$$

Want to prove:

$$\sum_{m\geq 0} m^{n+1} x^m = \frac{A_{n+1}(x)}{(1-x)^{n+2}}$$

So need:

$$A_{n+1}(x) = x(1-x)A_n'(x) + (n+1)xA_n(x)$$

Proposition 1.8. *For each $n \geq 0$,*

$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{m\geq 0}(m+1)^n t^m$$

Proof. I personally would leave this as an exercise with the hint to use the recursion given above. $\square$

Theorem 1.9.

$$\sum_{n\geq 0} A_n(t)\frac{x^n}{n!} = \frac{(1-t)e^{(1-t)x}}{1-te^{(1-t)x}}$$

Proof. Multiply both sides of Eq. (1) by $(1-t)^{n+1}x^n/n!$ and then sum from $n=0$ to infinity and we get

$$\begin{aligned}
\sum_{n=0}^{\infty} A_n(t)\frac{x^n}{n!} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(m+1)^n t^m (1-t)^{n+1}\frac{x^n}{n!} \\
&\overset{(a)}{=} \sum_{m=0}^{\infty} t^m \sum_{n=0}^{\infty}((m+1)(1-t))^n(1-t)\frac{x^n}{n!} \\
&\overset{(b)}{=} \sum_{m=0}^{\infty} t^m (1-t)e^{(m+1)(1-t)x} \\
&\overset{(c)}{=} (1-t)\sum_{m=0}^{\infty} t^m e^{m(1-t)x}e^{(1-t)x} \\
&\overset{(d)}{=} \frac{(1-t)e^{(1-t)x}}{1-te^{(1-t)x}}
\end{aligned}$$

switch order of summation

use identity $e^y = \sum_{j\geq 0} \frac{y^j}{j!}$

identity $e^{a+b} = e^{ab}$

formula for a geometric series

where

- In (a) we switched the order of summation.
- In (b) we used the identity $e^y = \sum_{j\geq 0}\frac{y^j}{j!}$.
- In (c) we used the identity $e^{a+b} = e^a e^b$ and moved $(1-t)$ outside the sum as it doesn't depend on m. (Same will be true for $e^{(1-t)x}$ in next step)
- In (d) we used the formula for a geometric series with common ratio $te^{(1-t)x}$

$\boxed{\text{40mins}}$ 18:50 ?

$\square$

PART B - 7.2 Descents and length generating functions

The most important statistic about a Coxeter group is its descent number

$$d(v) = |\{t \in S : \ell(vt) < \ell(v)\}|$$

Hence we get the following generating function

$$W(t;q) = \sum_{v\in W} t^{d(v)} q^{\ell(v)}$$

For example

$$A_2(t;q) = 1 + 2tq + 2tq^2 + t^2q^3$$
$$B_2(t;q) = 1 + 2tq + 2tq^2 + 2tq^3 + t^2q^4$$

} dihedral groups

Our aim is to obtain a simple recursive rule for computing the generating function for any Coxeter group W.

We can do this using the following theorem

Thm 7.2.1

$$\boxed{W(t;q) = \sum_{J \subseteq S} t^{|J|}(1-t)^{|S \setminus J|} \frac{W(q)}{W_{S \setminus J}(q)}}$$

Will not prove fully but a hint is that we need to rewrite

$$t^{d(v)} = \sum_{D(v) \subseteq J \subseteq S} t^{|J|}(1-t)^{|S \setminus J|}$$

from the generating function above.

Example on next page ↓

Theorem 7.2.1

$$W(t;q) = \sum_{J \subseteq S} t^{|J|}(1-t)^{|S \setminus J|} \frac{W(q)}{W_{S\setminus J}(q)}. \tag{7.10}$$

- No proof

Example

Ex 1: $W = S_2 = A_1$ $\quad \overset{\bullet}{s}$ "21", $J = \phi, \{s\}$

Thm $W = W_s$

$$\Rightarrow W_s(t;q) = t^{\overset{0}{|\phi|}}(1-t)^{\overset{1}{|\{s\}|}} \frac{\cancel{W_s(q)}}{\cancel{W_s(q)}} + t^{\overset{1}{|\{s\}|}}(1-t)^{\overset{0}{|\phi|}} \frac{W_s(q)}{W_\phi(q)}$$

$\overset{\bullet}{s}$ $W_s(q) = 1+q$

e → length 0

s → length 1

$$= (1-t) + t\,W_s(q) = (1-t) + t(1+q)$$

$$= 1 + qt$$

s = 21 ← 1 descent, length 1

e = 12 ← 0 descents, length 0

$$W(t,q) := \sum_{w \in W} t^{\overset{\text{descent}}{d(w)}} q^{\ell(w)} \leftarrow \text{length}$$

Ex 1: $W = S_2 = A_1$ $\quad \overset{\bullet}{s}$ "21" $\quad J = \emptyset, \{s\}$

Thm $W = W_s$

$$\Rightarrow W_s(t;q) = t^{|\emptyset|}(1-t)^{|\{s\}|} \frac{W_s(q)}{W_s(q)} + t^{|\{s\}|}(1-t)^{|\emptyset|} \frac{W_s(q)}{W_\emptyset(q)}$$

$\overset{\bullet}{s}$ $W_s(q) = 1+q$

$$(1-t) + t\,W_s(q) = (1-t) + t(1+q) = 1 + qt$$

However, there are sometimes better ways of computing $W(t;q)$ if W is a member of certain infinite families of Coxeter groups.

For example, $S_n(t;q)$ does not factor nicely

So instead we do

$$\sum_{n\geq 0} S_n(t;q)\frac{x^n}{[n]_q!} = \frac{(1-t)\exp(x(1-t);q)}{1-t\exp(x(1-t);q)}$$

References:

B. E. Sagan. Combinatorics: the art of counting. Vol. 210. Graduate Studies in Mathematics. https://users.math.msu.edu/users/bsagan/Books/Aoc/final.pdf. American Mathematical Society, Providence, RI, [2020] ©2020, pp. xix+304.

A. Bj ̈orner and F. Brenti. Combinatorics of Coxeter groups. Vol. 231. Graduate Texts in Mathematics. Springer, New York, 2005, pp. Xiv+363.

17. Eulerian numbers

Def

where $\exp(x;q) = \sum_{n\geq 0} \frac{x^n}{[n]_q!}$

equivalence class of n

If time:

Proof of Euler's formula for values of the alternating Ziemann zeta function

the function is $\varphi(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \ldots$

This converges for $Re(s) > 0$

and is related to the ζ - function
(zeta)

$$\varphi(s) = (1 - 2^{1-s})\,\zeta(s)$$

Euler wants to calculate $\varphi(-n)$ for $n = 0, 1, 2, 3 \ldots$

So he introduces the

Eulerian polynomials $P_n(t)$

$$\sum_{k=0}^{\infty} (k+1)^n t^k = \frac{P_n(t)}{(1-t)^{n+1}}$$

$tgh(x) \longrightarrow$ hyperbolic tangent

then $\varphi(-n) = P_n(-1)\,2^{-n-1}$

Using the eulerian polynomials we get

$$\sum_{n=0}^{\infty} P_n(t) \frac{x^n}{n!} = \frac{(1-t)e^{(1-t)x}}{1-te^{(1-t)x}}$$

hyperbolic tangent $= \tanh x = \frac{\sinh x}{\cosh x}$

$$= 1 + tgh(x)$$

Lastly, $\varphi(0) = \frac{1}{2}$ $\quad \varphi(-n) = \dfrac{tgh^{(n)}(0)}{2^{n+1}}$ for $n > 0$