Semi-Standard Young Tableaux

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4.2 Young Tableaux

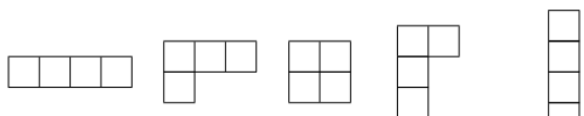
First we need to settle some definitions and notations regarding partitions and Young diagrams.

Definition 1. A **partition** of a positive integer n is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$ and $n = \lambda_1 + \lambda_2 + \cdots + \lambda_l$. We write $\lambda \vdash n$ to denote that λ is a partition of n.

For instance, the number 4 has five partitions: $(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)$. We can also represent partitions pictorially using Young diagrams as follows.

Definition 2. A **Young diagram** is a finite collection of boxes arranged in left-justified rows, with the row sizes weakly decreasing.[1] The Young diagram associated to the partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ is the one that has l rows, and λ_i boxes on the ith row.

For instance, the Young diagrams corresponding to the partitions of 4 are



(4) $(3, 1)$ $(2, 2)$ $(2, 1, 1)$ $(1, 1, 1, 1)$

Since there is a clear one-to-one correspondence between partitions and Young diagrams, we use the two terms interchangeably, and we will use Greek letters λ and μ to denote them.

A Young tableau is obtained by filling the boxes of a Young diagram with numbers.

Definition 3. Suppose $\lambda \vdash n$. A **(Young) tableau t of shape $\boldsymbol{\lambda}$**, is obtained by filling in the boxes of a Young diagram of λ with $1, 2, \ldots, n$, with each number occurring exactly once. In this case, we say that t is a λ-tableau.

For instance, here are all the tableaux corresponding to the partition $(2, 1)$:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$$

Definition 4. A **standard (Young) tableau** is a Young tableaux whose the entries are increasing across each row and each column.

The only standard tableaux for $(2, 1)$ are

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\,.$$

Here is another example of a standard tableau:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline 7 & 8 \\ \cline{1-2} 9 \\ \cline{1-1}\end{array}\,.$$

Example 1.6. There are 5 standard Young tableaux associated with the shape $(3,2)$. They are

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}\;,\quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}\;,\quad \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}\;,\quad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}\;,\quad \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}\;.$$

Definition 1.8. Given a shape $\lambda \vdash n$, a composition $\mu = (\mu_1, \ldots, \mu_k)$ and a **semistandard** Young tableau (SSYT) of λ is a filling of λ using content μ such that the followings hold:

- The number i appears μ_i times.
- All rows are weakly increasing.
- All columns are strongly increasing.

For instance, the semistandard tableau shown below may be seen to have shape $(4,2,1)$ and content $(2,2,1,0,1,1)$:

$$\begin{array}{|c|c|c|c|}\hline 1&1&2&5\\\hline 2&3\\\cline{1-2} 6\\\cline{1-1}\end{array}\;.$$

Definition 24. Let λ be a Young diagram. For a square u in the diagram (denoted by $u \in \lambda$), we define the **hook** of u (or at u) to be the set of all squares directly to the right of u or directly below u, including u itself. The number of squares in the hook is called the **hook-length** of u (or at u), and is denoted by $h_\lambda(u)$.

For example, consider the partition $\lambda = (5,5,4,2,1)$. The figure on the left shows a typical hook, and the figure on the right shows all the hook-lengths.

$$\begin{array}{|c|c|c|c|c|}\hline \;&\;&\;&\;&\;\\\hline &u&\bullet&\bullet&\bullet\\\hline &\bullet&&\\\cline{1-4} &\bullet\\\cline{1-2} \;\\\cline{1-1}\end{array}\qquad\qquad \begin{array}{|c|c|c|c|c|}\hline 9&7&5&4&1\\\hline 8&6&4&3&1\\\hline 6&4&2&1\\\cline{1-4} 3&1\\\cline{1-2} 1\\\cline{1-1}\end{array}$$

Theorem 25 (Hook-length formula). *Let $\lambda \vdash n$ be a Young diagram. Then*

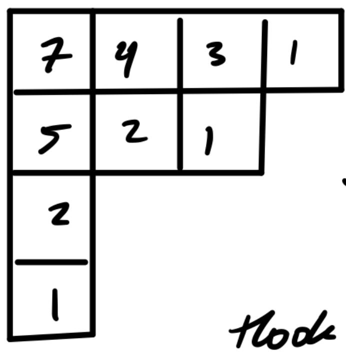
$$\dim S^\lambda = f^\lambda = \frac{n!}{\prod_{u\in\lambda} h_\lambda(u)}.$$

For instance, from the above example, we get

$$\dim S^{(5,5,4,2,1)} = f^{(5,5,4,2,1)} = \frac{17!}{9\cdot 8\cdot 7\cdot 6^2\cdot 5\cdot 4^3\cdot 3^2\cdot 2\cdot 1^5} = 3403400.$$

Ex.

Schur Polynomials



hook lengths for youngdiagram $\lambda = (4,3,1,1)$ corresponding to the partition $9 = 4+3+1+1$

hook-length gives the # of SYT as

$$f^\lambda = \frac{9!}{7\cdot5\cdot4\cdot3\cdot2\cdot2\cdot1\cdot1\cdot1} = 216$$

For $n \in \mathbb{Z}$, let $\mathrm{SSYT_n}(\lambda)$ be the set of all semistandard young tableau of shape λ with entries at most n.

Definition 1.2. The Schur polynomial associated to a partition λ is defined as

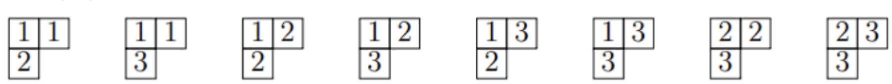
$$s_\lambda(x_1,\ldots,x_n) = \sum_{T\in\mathrm{SSYT_n}(\lambda)} x^{\vec{T}}$$

where $x^{\vec{T}} = x_1^{\alpha_1}\ldots x_n^{\alpha_n}$ where $\alpha_i = \#$ of times i appears in T.

Definition 1.3. A polynomial $f(x_1,\ldots,x_n)$ is symmetric if $f(x_1,\ldots,x_n) = f(x_{\pi(1)},\ldots,x_{\pi(n)})$ for all $\pi \in S_n$.

Theorem 1.4. *Let λ be a partition of n. Then $s_\lambda(x_1,\ldots,x_n)$ is a symmetric polynomial.*

Example 1.5. Here are the 8 different SSYT of shape $(2,1)$ with entries at most 3.



The corresponding schur polynomial is

$$s_{(2,1)}(x_1,x_2,x_3) = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$$

which is symmetric.

ex [2×1 column] [1×2 row] SSYT([2×1 column]) = {1 over 2} → $S_{(1,1)}(x_1, x_2) = x_1 x_2$

SSYT([1×2 row]) = { [1 | 2], [1 | 1], [2 | 2] } → $S_{(2)} = x_1^2 + x_1 x_2 + x_2^2$

both symmetric

Note: $\delta_n = (n, n-1, \ldots, 2, 1)$ is called a staircase partition.

Theorem 7.4.7 *The map $\widehat{p}$ is a bijection between the set of all tableaux of shape δ_n and $\mathcal{R}(w_0)$, where w_0 denotes the longest element of S_{n+1}.*

> Remind what reduced decomposition of $w \in W$

$R(w)$ = set of all reduced decompositions of w

coxeter group = < { s_i } | rules/relations >

→ an element $w \in W$ can potentially be written in different expressions in the generators

$S_3 = \langle s_1, s_2 \mid (s_1 s_2 s_1) = (s_2 s_1 s_2),\ s_1^2 = s_2^2 = e \rangle$

$\pi = 321$

→ thm 7.4.7 ∃ a bijection, $w_0 \in S_{n+1}$

$|R(w_0)| \longleftrightarrow |SYT(\delta_n)|$

ex. $n = 2$ $|R(321)| \longleftrightarrow |SYT([shape (2,1)])|$ ← δ_2

$\pi = 321 = sts = tst$ ⟷ (staircase partition)

↕

[shape (2,1)] [shape (2,1)]

1 2   1 3
3   2

Corollary 7.4.8

$$|\mathcal{R}(w_0)| = \frac{\binom{n+1}{2}!}{1^n 3^{n-1} 5^{n-2} \cdots (2n-1)}.$$

Proof. This follows immediately from Theorem 7.4.7 and the well-known "hook length formula" (see, e.g., [498, Corollary 7.21.6]). □

2. Preliminaries

Given a Young diagram, the *hook* at any one of its boxes is the collection of boxes to the right and below that box, and including the box itself. We write "ℓhook" for the *length* of such a hook, i.e. for the number of boxes it contains. Formally:

Definition 2.1. (hook length)

Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{\mathrm{rows}(\lambda)})$ be a partition/Young diagram. Then for

- $i \in \{1, \cdots, \mathrm{rows}(\lambda)\}$,
- $j \in \{1, \cdots, \lambda_i\}$

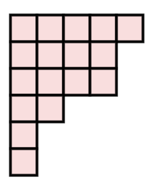
the *hook length* at (i, j) is

$$\ell\mathrm{hook}(i, j) \;\coloneqq\; 1 + (\lambda_i - j) + (\lambda'_j - 1),$$

where λ' denotes the *conjugate partition* (see there).

Conjugate Partition:

partition (5, 4, 4, 2, 1, 1) of 17 is given in the English representation as:



Let $\mathbb{Y}$ be the set of Young diagrams. Important functions on Young diagrams include:

- ***conjugation***: denoted by a prime $' : \mathbb{Y} \to \mathbb{Y}$ reflects the Young diagram along its main diagonal (north-west to south-east). In the above example the conjugated partition would be $\lambda' = (6, 4, 3, 3, 1)$.

- $n \in \mathbb{N}$ a natural number;
- $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{\mathrm{rows}(\lambda)})$ a partition of n, $\sum_i \lambda_i = n$, equivalently a Young diagram with n boxes;
- $N \in \mathbb{N}_+$ a positive natural number;

Write:

- $\mathrm{ssYT}_\lambda(N)$ for the set of semistandard Young tableaux T
  - of shape (i.e. with underlying Young diagram) λ,
  - with labels $T_{i,j} \leq N$ (i.e. with $T_{i,j} \in \{1, \cdots, N\}$).
- $n(T) = \sum_{i,j} T_{i,j}$ for the sum of all the labels.

3. Statement

Standard hook-content formula

***Proposition 3.1**. For λ a partition/Young diagram and $N \in \mathbb{N}_+$, the number $|\mathrm{ssYTableaux}_\lambda(N)|$ of semistandard Young tableaux T of shape λ (hence the value of the Schur polynomial s_λ on N unit argument) and entries bounded by $T_{i,j} \leq N$ is:*

$$|\mathrm{ssYTableaux}_\lambda(N)| \;=\; s_\lambda\left(x_1 = 1, \cdots, x_N = 1\right) \;=\; \prod_{(i,j)} \frac{N + \mathrm{content}(i,j)}{\ell\mathrm{hook}_\lambda(i,j)}.$$

q-Deformed hook-content formula

With the above ingredients, we have the following equalities of polynomials in a variable q:

$$\sum_{T \in \mathrm{ssYT}_\lambda(N)} q^{n(T)} \;=\; q^{\sum_i i \cdot \lambda_i} \cdot \prod_{\substack{i \in \{1, \cdots, \mathrm{rows}(\lambda)\} \\ j \in \{1, \cdots, \lambda_j\}}} \frac{1 - q^{N + \mathrm{content}(i,j)}}{1 - q^{\ell\mathrm{hook}(i,j)}}$$

(Krattenthaler 98, Thm. 1)

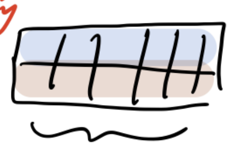
Compute an explicit hook-length formula for the partition (n,n)

d. Hook-length formula

1. $\#|SYT(\lambda)| = \frac{n!}{\pi \text{ hooks}}$

cardinality, number, how many

ex.



n boxes

(n,n) is a partition of 2n

$$\to \frac{(2n)!}{((n+1)(n-1+1)\dots 2)(n(n-1)\dots 1)}$$

$$= \frac{1}{n+1}\frac{(2n)!}{n!n!}$$

$$= \frac{1}{n+1}\binom{2n}{n}$$

$$\binom{a}{b} = \frac{a!}{b!(a-b)!}$$