The Bruhat Order of S_n

Savik Kinger

October 13, 2022

Abstract

In this week's lecture, we will present the Bruhat Order for Coxeter Systems, specifically focusing on the Bruhat Order for the Symmetric Group. We will show how the Bruhat Order allows us to establish a poset structure on the Symmetric Group. We will also show how we can deterministically decide if two elements in the Symmetric Group are comparable with respect to the Bruhat Order. We loosely follow Chapter 2.1 of Combinatorics of Coxeter Groups.

Contents

1 Motivation and Review **1**

2 Bruhat Order and Graph **2**

3 Bruhat Order on the Symmetric Group **3**

4 The Comparable Problem **5**

5 History of the Bruhat Ordering **6**

1 Motivation and Review

Recall the definition of Poset a previous talk.

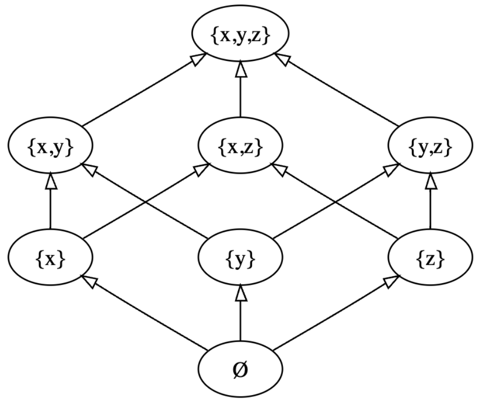
Definition 1. A Partially Ordered Set (poset) is a set S with an ordering $<$ such that

1) Every element is related to itself

2) For $x, y \in S$, if $x < y$ and $y < x$ then $x = y$. Notice a consequence is that if $x \neq y$ we have that either $x < y$, $y < x$ or y is not related to x

3) For $x, y, z \in S$ we have $x < y, y < z \implies x < z$

Recall the construction of Hasse Diagrams for posets. An example is provided for the set $\{x, y, z\}$ with the subset ordering:



We want to impose this type of geometric structure for Coxeter Systems. This sets up the construction of the Bruhat Order and subsequent Bruhat Graph.

2 Bruhat Order and Graph

For the following examples, we assume (W, S) is a Coxeter System and define $T = \{wsw^{-1} : w \in W,\, s \in S\}$ to be (W, S) corresponding set of reflections.

Definition 2. Let $u, w \in W$, then we have the following:

1) Define $u \xrightarrow{t} w$ to mean that $w = tu$ for the specified $t \in T$ and $l(u) < l(w)$ where l denotes the length of the word.

2) Define $u \to w$ to mean that $u \xrightarrow{t} w$ for some arbitrary $t \in T$.

3) Define $u \leq w$ to mean there exists some $w_i \in W$ such that:

$$u = u_0 \to u_1 \to \ldots \to u_{k-1} \to u_k = w$$

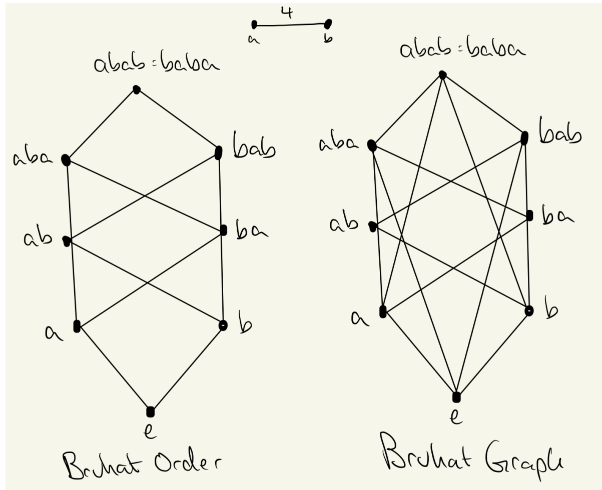
Some intuition follows from the fact that our ordering depends on there being reflections which increase the size of the word. Geometrically, we can think of this as saying that the reflection is occuring on a different axis from some base configuration.
The Bruhat order defined above is the partial order relation on the set w defined by the ordering (3) from the definition.

Remark 1. We notice the following from the definition

- $u < w$ implies $l(u) < l(w)$
- $u < ut$ holds iff $l(u) < l(ut)$ forall $u \in W$ and $t \in T$
- The identity element satisfies $e \leq w$ for all $w \in W$. Since we have that any $w = s_1 s_2 \ldots s_k$ so we can order $e \to s_1 \to s_2 \to \ldots s_k$

Definition 3. The Bruhat Graph is the Hasse Diagram induced on a Coxeter System under the Bruhat Ordering. We add the edges corresponding to the second condition of Definition 1 as well.

Example 1. Consider Coxeter System defined for the group dihedral group B_2. We have $T = \{a, b, aba, bab\}$ and we show the Bruhat Order and Graph below:



3 Bruhat Order on the Symmetric Group

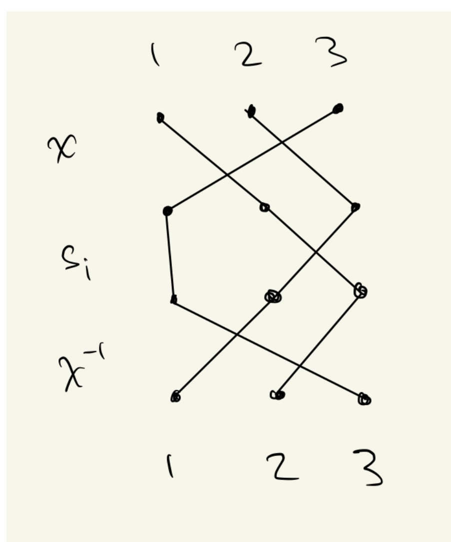
Recall from Chapter 1 that S_n, the Symmetric Group, is a Coxeter Group with respect to the generating set of adjacent transpositions $s_i = (i, i+1)$. We define the reflection set of S_n as

$$T = \{(a, b) \,|\, 1 \leq a < b \leq n\}$$

which follows if we consider that a reflection in S_n around the generating element s_i is

$$xs_ix^{-1} = (x(i), x(i+1))$$

Example 2. For intuition, we can consider a wiring diagram to describe the permutation:



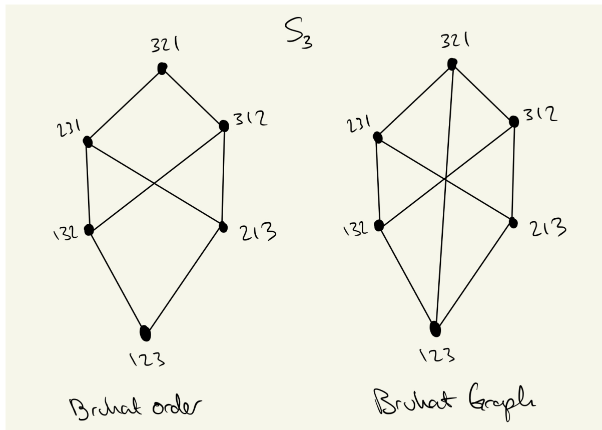
Notice that we have that the act of conjugation is changing where the transposition occurs, but it remains a transposition.

By reflections t in S_n being transpositions and lengths of element x are determined by the number of inversions. We have the following remark.

Remark 2. The relation $x \xrightarrow{(a,b)} y$ implies the permutation $x(1)\dots x(n)$ going to the permutation y occurs by transposing the positions of $x(a)$ and $x(b)$ where $a < b$ and $x(a) < x(b)$.

These transpositions describe the edges of the Bruhat Graph for S_n

Example 3. Consider Coxeter System defined for the group (S_3, S) with the generating set $S = \{(12), (23)\}$. We show the Bruhat Graph below:



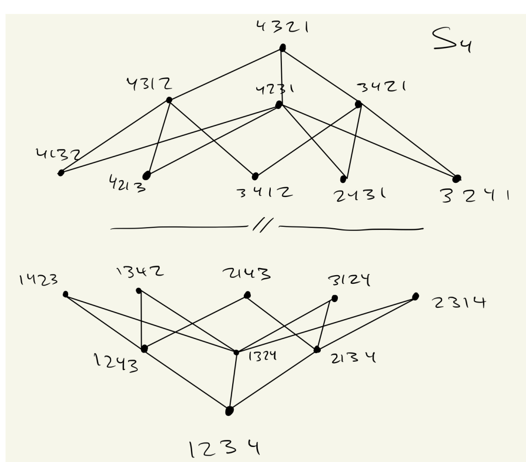
We use this observation to motivate the following lemma.

Lemma 1. *Let $x, y \in S_n$. We have x is covered by y in Bruhat Order iff $y = x \cdot (a, b)$ for some $a < b$ such that $x(a) < x(b)$ and there does not exist any c such that $a < c < b$, $x(a) < x(c) < x(b)$ holds.*

Proof. If $y = x \cdot (a, b)$ for some (a, b) where $a < b$, then we have that $y \geq x$ by definition. So we then have that $y = xt \implies inv(y) = inv(x) + 1$, where inv denotes the inversion number. This implies we have a Bruhat Ordering. We now show the converse. For y to cover x, we want $l(y) = l(x) + 1$. Then we know $x(a) < x(b)$. So if $x(a) < x(c) < x(b)$ for some $a < c < b$, we have $x < x \cdot (a, c) < y$ so $x < y$ cannot be a covering. As that would imply there is an extra transposition beyond the one added in the formation of y from x. As by assumption we know that is not the case, we have that y does cover x. ■

The benefit of this lemma is that for sufficently small n, we can determine whether two elements are comparable. That is, are they ordered and if so, how?

Example 4. When $n = 4$, the lemma allows us to construct an ordering on S_4 with the Bruhat Graph:



4 The Comparable Problem

A natural question that arises under our partial ordering is if we given two elements, do we know if they are Comparable?

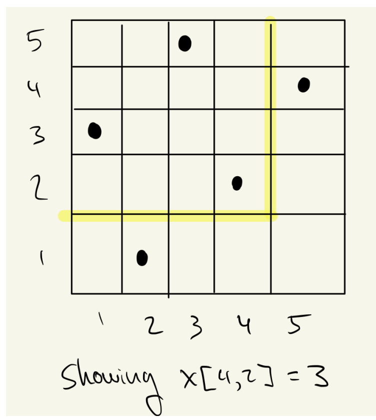
Example 5. For $x, y \in S_10$ where $x = 368475912$ and $y = 694287531$, are x and y comparable? (For general groups, this problem is undecidable)

We now present a method for answering this question. We first introduce the following notation. For $x \in S_n$, let

$$x[i, j] = |\{a \in [i] \, : \, x(a) \geq j\}|$$

for $i, j \in [1, n]$ An interpretation of this notation can be thought of plotting the element x on a $n \times n$. We plot such points by taking for some $a \in [1, n]$, plotting the point $(a, x(a))$ on the grid and counting the number of points above and to the left of the point $(a, x(a))$.

Example 6. We provide an example of this process for the permutation $x \in S_5$ where $x = 31524$. We notice that $x[1, 3] = 1$, $x[3, 3] = 2$ and so on.



From this picture, we arrive at the following observations:

Remark 3. For any $x \in S_n$

- $x[n, i] = n + 1 - i$ and $x[i, 1] = i$ for $i \in [1, n]$
- $x[i, j] - x[k, j] + x[k, l] = |\{a \in [k + 1, i] \, : \, j \leq x(a) < l\}|$ for all $1 \leq k \leq i \leq n$ and $1 \leq j \leq l \leq n$. This follows from the above counting dot representation.

Theorem 1 (2.1.5)**.** *Let $x, y \in S_n$, then TFAE*

1) $x \leq y$

2) $x[i, j] \leq y[i, j] \; \forall i, j \in [1, n]$

This theorem allows us to deterministically determine if two elements in S_n are ordered according to the Bruhat Order, since we need only check at most all $i, j \in [1, n]$ which is on the order of n^2 computations.

Proof. Suppose that (1) holds. Then we may assume that $x \to y$. This means by definition that there exists $1 \leq a < b \leq n$ such that $y = x \cdot (a, b)$ and $x(a) < x(b)$. By definition of $x[i, j]$ we arrive at:

$$y[i, j] = \begin{cases} x[i, j] + 1 & \text{for } a \leq i < b, \, x(a) < j \leq x(b) \\ x[i, j], & \text{otherwise} \end{cases}$$

We have from the above that the second condition follows directly.

We now show the converse. Suppose that (2) holds. Define $M(i,j) = y[i,j] - x[i,j]$ for all $i, j \in [1,n]$. If $M[i,j] = 0$ for all $i, j \in [1,n]$ then $x = y$. Let $(a_1, b_1) \in [1,n]^2$ be such that $M(a_1, b_1) > 0$ and $M[i,j] = 0$ for all $(i,j) \in [1, a_1] \times [b_1, n]/\{(a_1, b_1)\}$. Then $y(a_1) = b_1$ and $x(a_1) < b_1$. Now, let $(a_2, b_2) \in [1,n]^2$ be the bottom right corner of a maximal positive connected submatrix of M having (a_1, b_1) as the upper left corner. It follows from remark 3 (1) on $x[n,i]$ that $a_2 < n$ and $b_2 > 1$. Because of maximality, there exists $c \in [a_1, a_2]$ and $d \in [b_2, b_1]$ such that $M(c, b_2 - 1) = 0$ and $M(a_2 + 1, d) = 0$. Hence,

$$M(a_2 + 1, b_2 - 1) - M(c, b_2 - 1) - M(a_2 - 1, d) + M(c, d) > 0$$

By the remark 3 (2), we see that this simplifies to

$$\{e \in [c+1, a_2+1] \,:\, y(e) \in [b_2 - 1, d - 1]\}| > 0$$

So let $(a_0, b_0) \in [c+1, a_2+1] \times [b_2 - 1, d - 1]$ be such that $y(a_0) = b_0$. Then, we have have that $a_1 < a_0$ and $y(a_1) = b_1 > b_0 = y(a_0)$ It follows then that $z \to y$, where $z = y \cdot (a_1, a_0)$ However, because $x[i,j] \leq z[i,j]$ for all $i, j \in [1,n]$ by our definition of $y[i,j]$ and our choice of a_2, b_2, by induction we get that $x \leq z$ and thus $x \leq y$. ■

An example of this theorem, as shown in our question of determining whether $x = 368475912$ and $y = 694287531$ are comparable, is from the observation that $x[1,6] < y[1,6]$ and $x[4,3] > y[4,3]$ so it follows that the two are not comparable.

5 History of the Bruhat Ordering

Origins arrive from Hilbert's 15 Problem from a list of 70 problems that he believed to be some of the most important problems of his era. (Circa 1700)
The Schubert calculus is concerned with the questions of the form:

- How many lines go through 2 points?
- How many conics go through 3 points?
- How many surfaces go through 3 curves?

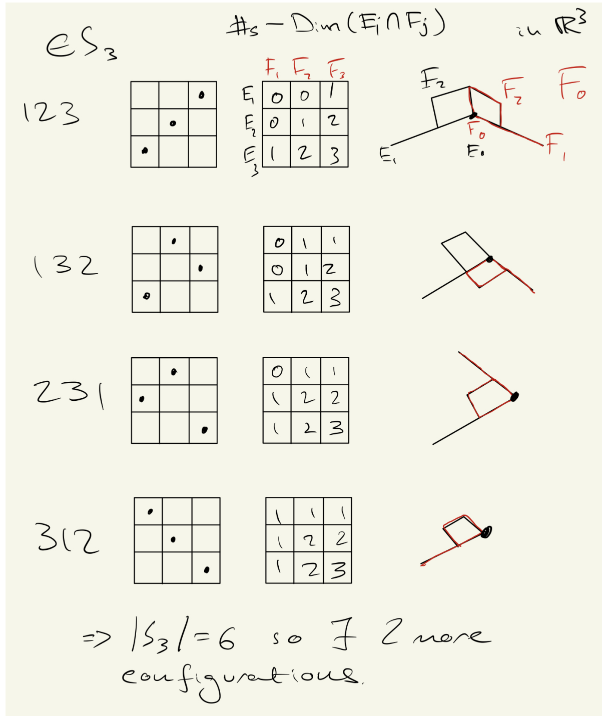
It turns out we can use algebra to solve this problem. The numbers involved in how many points and the dimensions of the shapes involved show up as multiplication constants of polynomials in a cohomology ring. So solving this problem comes down to solving for constants in a polynomial, whose solution can be found by using the Bruhat Order to rewrite this polynomial in terms of the reflection-based ordering that the Bruhat Order induces.
We provide some intuition by considering a case of the Schubert calculus that is concerned with the questions of how flags intersect. [1]
Define a *Flag* in $\mathbb{R}^n$ to be

$$F \,:\, 0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{R}^n$$

Where the dimension of $F_i = i$. We can think of a flag in $\mathbb{R}^3$ as a point when $i = 0$, when $i = 1$ it is a line and it is a plane when $i = 2$ and so on.

[1]See Lecture

We arrive at the following structure:



The matricies arrive from considering the dimension of the intersections at each of the points. The Dotted matricies are generalizations of the numerical matricies such that the number of each matrix corresponds to the number of dots above and to the left of the numbers. And then the dotted matricies correspond to permutation matrices which correspond to configurations of 3 elements. Notice that as we go up we increase in ordering in the group S_3. So what this example is illustrating is how we can use the structure of the Bruhat order geometrically to describe how structures intersect.