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The Combinatorics of Coxeter Groups
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Partially Ordered Sets (Posets) and Lattices

I. Definitions

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a) Partially Ordered Sets
- **Definition:** A partially ordered set or poset is a pair $(P, \leq)$ where "P" is a set and "$\leq$" is a binary relation on P satisfying the following axioms for all $x, y, z \in P$:
  i) $x \leq x$ (reflexivity)
  ii) if $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry)
  iii) if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity)

- **Example:** You can tell $(Z, \neq)$ is not a poset immediately because it's not reflexive.

b) Lattices
- **Definition:** Lattices are an important class of partially ordered sets with the property that pairs of elements have greatest lower bounds and least upper bounds.

II. Working with Posets

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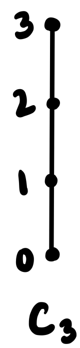
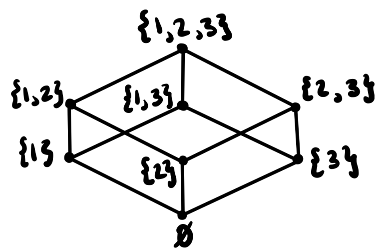
a) Drawing out Posets
- The Hasse (Ha-suh) diagram of P is the graph with vertices P and an edge from x up to y if $x \lessdot y$.
- **Definition:** For $x \in P$ we say that x is covered ($\lessdot$) by y, if $x < y$ and there is no $z \in P$ with $x < z < y$.

- C_n: The chain of length n is $(C_n, \leq)$ where $Cn = \{0, 1, \ldots, n\}$ and $i \leq j$ is the usual ordering of the integers.
  - **Example:** C_3

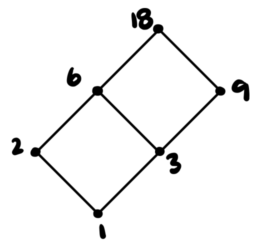


- B_n: The Boolean algebra is $(B_n, \subseteq)$ where $B_n = 2^{[n]}$ and $S \subseteq T$ is set containment.
  - **Example:** B_3



B_3

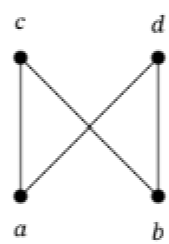
- D_n: The divisor lattice is(D_n, |) where D_n consists of all the positive integers which divide evenly into n and a | b means that a divides evenly into b (in that b/a is an integer).
  - **Example:** in D_{12} we have $2 \leq 6$ but $2 \nleq 3$
  - **Example:** D_{18}



D_{18}

b) Minimal and Maximal Elements

- **Definition:** A minimal element of P is x $\in$ P such that there is no y $\in$ P with y < x. Note that P can have multiple minimal elements. Same true for maximal.
  - **Example:** 2 minimal elements below, a and b.

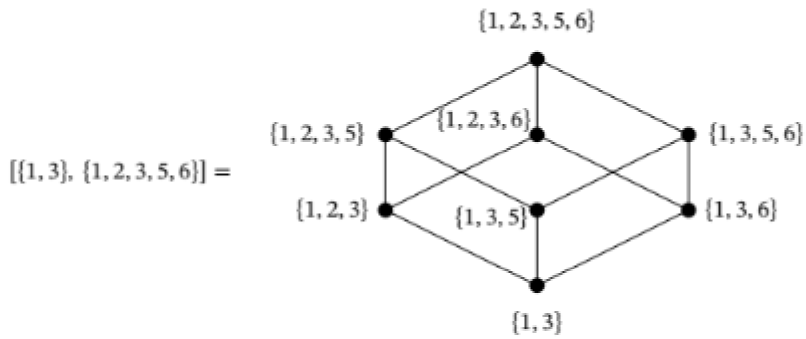


- **Definition**: P has a minimum element if there is x $\in$ P such that x $\leq$ y for every y $\in$ P.
- **Claim:** A minimum element is unique if it exists because if x and x′ are both minimum elements, then x $\leq$ x′ and x′ $\leq$ x which forces x = x′ by antisymmetry.
  - In this case the minimum element is often denoted $\hat{0}$.
- In the daul case the maximum element is x $\in$ P which satisfies x $\geq$ y for all y $\in$ P.
  - A maximum is unique if it exists and is denoted $\hat{1}$
- It's helpful to think of the actual Latin. "Minimum" means "smallest" and "Minimal" means "nothing is smaller."

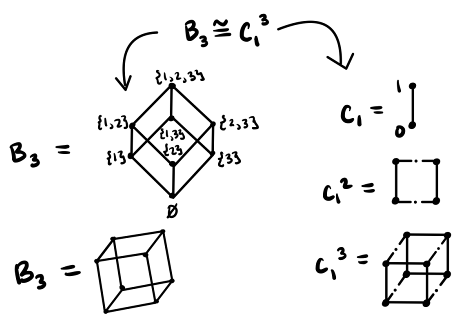
- **Proposition 5.1.1** We have the following minimum and maximum elements:

- In C_n we have $\hat{0} = 0,\ \hat{1} = n$
- In B_n we have $\hat{0} = \emptyset,\ \hat{1} = [n]$
- In D_n we have $\hat{0} = 1,\ \hat{1} = n$

c) Subposets
- We can learn more about poset P by looking at its substructures.
- **Definition:** A subposet of P is a subset $Q \subseteq P$ with the inherited partial order; namely $x \leq y$ for $x, y \in Q$ if and only if $x \leq y$ in P.
- **Example:** Assume $x, y \in P$. Then the corresponding closed interval is $[x, y] = \{z \in P \mid x \leq z \leq y\}$.
- Note that $[x, y] = \varnothing$ unless $x \leq y$.
- **Example**: the Hasse diagram of the interval $[\{1, 3\}, \{1, 2, 3, 5, 6\}]$ in B_7.



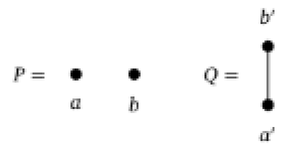
d) Products of Posets
- We can also produce posets from from old ones via products.
- Given two (not necessarily disjoint) posets $(P, \leq_P)$ and $(Q, \leq_Q)$, their (direct or Cartesian) product has underlying set $P \times Q = \{(x, y) \mid x \in P, y \in Q\}$ together with the partial order $(x, y) \leq P{\times}Q\ (x', y')$ if $x \leq_P x'$ and $y \leq_Q y'$.
- One can obtain the Hasse diagram for $P \times Q$ by replacing each vertex of Q with a copy of P and then, for each edge between two vertices of Q, connecting each pair of vertices having the same first coordinate in the corresponding two copies of P.
- **Proposition 5.2.1** We have the following product decompositions:
  a) $B_n \cong C_1^n$
  b) If the prime factorization of n is $n=p_1^{n1}\ p_2^{n2} \ldots p_k^{nk}$, then $D_n \cong C_{n1} \times C_{n2} \times \ldots \times C_{nk}$
- **Example:** $B_3 \cong C_1^3$, isomorphic, same Hasse diagram.



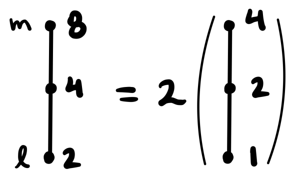
III. Observing Isomorphism

- Start by considering maps on posets.

- **Definition** Assume posets P, Q. Then a function f: P → Q is order preserving if $x \leq_P y \Rightarrow f(x) \leq_Q f(y)$
- **Example:** The map f : Cn → Bn by f(i) = [i] is order preserving because $i \leq j$ implies that
  $f(i) = [i] \subseteq [j] = f(j)$.
- We say that f is an isomorphism or that P and Q are isomorphic, written $P \cong Q$, if f is bijective (and both f and f^{-1} are order preserving).
  - It is important to show that f^{-1}, and not just f, is order preserving.
- **Example:** Two non-isomorphic posets.



- **Proposition 5.1.3** We have the following important isomorphism:
  - If $l, m \in D_n$ then $[l, m] \cong D_{m/l}$
- **Example:** if $l=2$, $m=8 \in D_n$ then $[2, 8] \cong D_{8/2}$



- If the Hasse Diagrams are the same then the two posets are isomorphic.

IV. Working With Lattices

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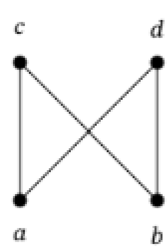
a) More on Lattices
- Lattices are an important class of partially ordered sets.
- **Definition:** If P is a poset and $x, y \in P$, then a lower bound for x, y is a $z \in P$ such that $z \leq x$ and $z \leq y$.
- **Example:** if $S, T \in Bn$, then any set contained in both S and T is a lower bound.
- We say that x, y have a greatest lower bound or meet if there is an element in P, denoted $x \wedge y$, which is a lower bound for x, y, and $x \wedge y \geq z$ for all lower bounds z of x and y.
- Returning to B_n, we have $S \wedge T = S \cap T$.
  - If the meet of x, y exists, then it is unique.
  - If z, z′ are both greatest lower bounds of x, y, then we have $z \geq z'$ since z′ is a lower bound and z is the greatest lower bound.
  - But interchanging the roles of z and z ′ also gives $z' \geq z$. So $z = z'$ by antisymmetry.
- Note also that it is possible for the meet not to exist.
- **Example:** in the poset of the figure below, $a \wedge b$ does not exist because this pair has no lower bound. Also, $c \wedge d$ does not exist but this is because the pair has both a and b as lower bounds, but there is no lower bound larger than both a and b.



- These definitions can be extended from pairs of elements to any nonempty set of elements X = {x1 , ... , xn} $\subseteq$ P where the meet is denoted:

$$\bigwedge X = \bigwedge_{x \in X} x.$$

- The only element y of P such that x $\wedge$ y = x for all x is y = $\hat{1}$.
- The concepts of upper bound and least upper bound are obtained by reversing the inequalities in the definitions of the previous paragraph.
- If the least upper bound of x, y exists, then it is denoted x $\vee$ y and is also called their join.
- **Key Point:** A lattice is a poset such that every pair of elements has a meet and a join.

b) Terminology and Properties

Proposition 5.3.1. *The posets $C_n, B_n, D_n, \Pi_n, Y, K_n$, and $L(V)$ are all lattices for all n and V of finite dimension over some $\mathbb{F}_q$. In addition, we have the following descriptions of their meets and joins.*

(a) *If $i, j \in C_n$, then $i \wedge j = \min\{i, j\}$ and $i \vee j = \max\{i, j\}$.*

(b) *If $S, T \in B_n$, then $S \wedge T = S \cap T$ and $S \vee T = S \cup T$.*

(c) *If $c, d \in D_n$, then $c \wedge d = \gcd(c, d)$ and $c \vee d = \operatorname{lcm}(c, d)$.*

Proposition 5.3.2. *Let L be a lattice. Then the following are true for all $x, y, z \in L$.*

(a) (*Idempotent law*) $x \wedge x = x \vee x = x$.

(b) (*Commutative law*) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$.

(c) (*Associative law*) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$.

(d) (*Absorption law*) $x \wedge (x \vee y) = x = x \vee (x \wedge y)$.

(e) $x \leq y \iff x \wedge y = x \iff x \vee y = y$.

(f) *If $x \leq y$, then $x \wedge z \leq y \wedge z$ and $x \vee z \leq y \vee z$.*

References

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2) Hasse Diagrams of Integer Divisors - Wolfram Demonstrations Project

3) PARTIALLY ORDERED SETS, CMU Mathematics

4) Lec 16 2 Join,Meet, and All About Lattice