Matroids + the work of June Huh

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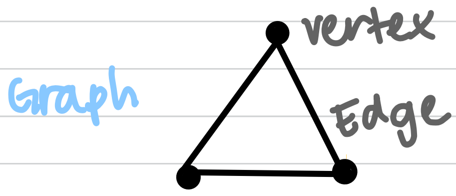
consider a simple graph a triangle



Mathematicians are interested in the following question:
How many different ways can you color the vertices of the triangle given some number of colors and adhering to the rule that whenever two vertices are connected by an edge they can't be the same color?

If you have q colors:
1. you have q options for the first vertex
2. q-1 options for the adjacent vertex because you can use any color save the color you used to color the first vertex
3. q-2 options for the third vertex because you can use any color save the two colors you used to color the first two vertices

Total # of colorings: $q \times (q-1) \times (q-2) = \boxed{q^3 - 3q^2 + 2q}$

This equation is called the chromatic polynomial for the graph

It has interesting properties
1. sequence is unimodal: The sequence peaks once, before that rises and falls only after

For a triangle: $1q^3 - 3q^2 + 2q$

$\downarrow$

1, 3, 2 (absolute value of the sequence)

Unimodal 1, 3, 2
peak

other examples of unimodal sequences:
1, 2, 3, 4, $\boxed{5}$, 4, 3, 2, 1
2, 3, 5, 7, $\boxed{9}$, 8, 7, 6, 5

2. The sequence is "log concave" meaning that any three consecutive numbers in the sequence follow this rule: the product of the outside two numbers is less than the square of the middle number

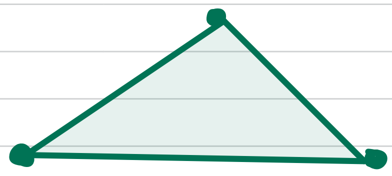
(1, 3, 2) is log concave ($1 \times 2 = 2 < 3^2$)
(2, 3, 5) does not ($2 \times 5 = 10 > 3^2$)

The fact that these two properties always holds is called Read's conjecture
↳ June Huh, the focus of the talk today proved this

consider a slightly more complicated graph, a rectangle

we can calculate the chromatic polynomial by breaking up a graph into subgraphs
subgraphs are all the graphs you can make by deleting an edge (or edges) from the original graph or by contracting two vertices into one

rectangle with deleted graph



rectangle with contracted edge

Chromatic polynomial of the rectangle is equal to the chromatic polynomial with one edge deleted minus the chromatic polynomial of the triangle

rectangle with deleted graph

$q^4 - 3q^3 + 3q^2 - q$

—

rectangle with contracted edge

$q^3 - 3q^2 + 2q$

chromatic polynomial of rectangle

$= q^4 - 4q^3 + 6q^2 - 3q$

Log concavity is not always preserved with addition/subtraction but with chromatic polynomials it is

MATROIDS: Graphs are one type of object that can define a more general structures called ~~ma~~ matroids.
Consider for example, two points on a two-dimensional plane. If more than two points lie on a line in this same plane, you can say those points are dependent.
Matroids are abstract concepts that capture notions like dependence and independence in all sorts of different contexts — from graphs to vector spaces to algebraic fields

Matroids associated with graphs:

we define a matroid M(G) associated with the graph G by specyfying the ground set and the independent set.
A subset of sets is called acrylic if it contains no ~~dect~~ cycles

let E be a finite set and let I be a family of subsets of E. Then the family I forms the independent sets of a matroid M if:

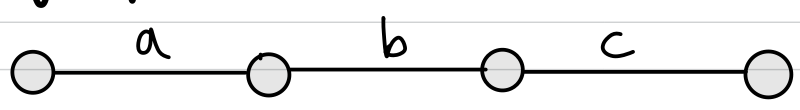
1) $I \neq \emptyset$ 2) $J \in I$ and $I \subseteq J$ then $I \in J$
3) If $I, J \in I$ with $|I| < |J|$ then there is some element $x \in J - I$ with $I \cup \{x\} \in I$

E is called the ground set of the matroid

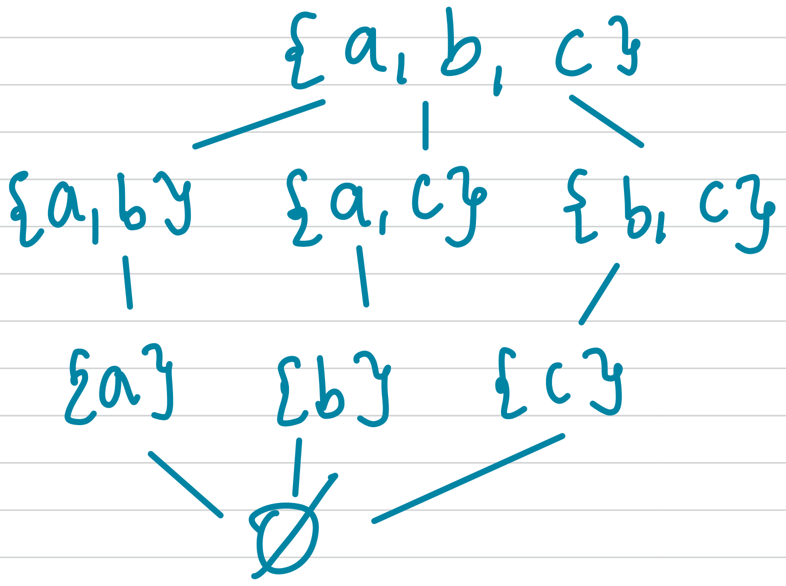
→ write out def of independent/acrylic sets

Theorem 4.1: let E be the set of edges of a graph and let $\mathcal{I}$ be the collection of all subsets of edges that are acrylic. Then $M = (E, \mathcal{I})$ is a matroid

Example: we are going to compute the matroid associated with the graph



1. write out all independent sets



$= U_{3,3} = B_3$

matroid $U_{n,n}$ is called the Boolean Algebra

$U_{k,n}$ **is a uniform matroid:** matroid in which the independent sets are exactly the sets containing at most r elements, for some fixed integer

$n =$ size of matroid

$k=2$ refers to the fact that every subset of E ~~to~~ that has 2 or fewer subsets is independent

In general $U_{k,n}$ is a matroid with $|E| = n$ and every subset of E with k or fewer elements is independent

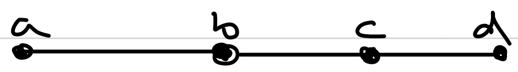
Now, lets talk about the relationship between matroids and graphs.

First important question: **Do all matroids come from graphs?** **More precisely,** can we always find a graph G with $M(G) = M$?

(This means the independent sets of the matroid M must precisely match the cycle-free subsets of edges of G. the satisfying answer to this question is No

Matroids that do arise as cycle matroids are called $\boxed{\text{graphic}}$

Example 1.19: The uniform matroid $U_{2,4}$ in Figure 1.25 is not graphic



All graphic matroids are representable

(representable definition: a matroid whose ground set E is a set of vectors)

RANK: Given any subset A of the ground set E of the matroid we can look at the size of all independent sets that are contained in A. The largest such independent subset of A is its rank

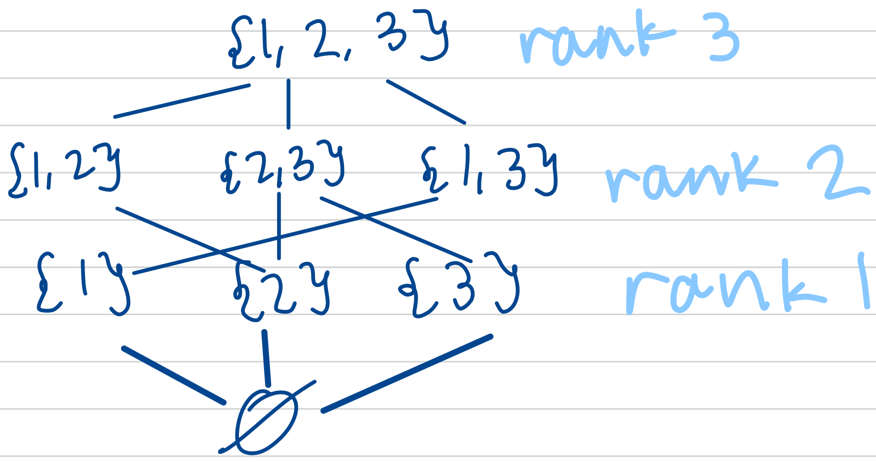
Definition 2.1.2: let $M = (E, \mathcal{I})$ be a matroid and let $A \subseteq E$. the rank of A, written $r(A)$ is the size of the the largest independent subset of A

$$r(A) := \max_{I \subseteq A} \{ |I| : I \in \mathcal{I} \}$$

compute the ranks for $B_n = U_{n,n}$, the boolean algebra and $U_{2,4}$

$B_n = U_{n,n}$

The rank of a boolean algebra $B_n = n$



Everything is a flat

Next concept comes from geometry. A flat in a matroid is a subset that is rank-maximal: if you add anything new to a flat, its rank increases

Definition 2.16: Let E be the ground set of the matroid M. A subset $F \subseteq E$ is a flat if $r(F \cup \{x\}) > r(F)$ for any $x \notin F$

A set has rank 2 if it spans a line in the geometry. Among the points a,b,c and d, choosing any two (or more) except ab will span the line con- taining these four points.

Theorem 6.1: Given a matroid M, let $(\mathcal{F}(M), \subseteq)$ be the poset where $\mathcal{F}(M)$ is the set of all flats of M and $\subseteq$ is set inclusion. Given $(L, \leq)$ a lattice, TFAE
the following are equivalent

1) $(L, \leq)$ is a geometric lattice (atomic and rank function is semimodular)

2) $(L, \leq)$ is isomorphic as a poset to $(\mathcal{F}(M), \subseteq)$ for some matroid M.

d) Characteristic Polynomials Pt. 2

→ one way to get insight into a combinatorial object is to study its generating function let's define the characteristic polynomial of a graded poset

Definition 5.1.5: Let P be a finite ranked poset with $rk\, P = n$. The characteristic polynomial of P is

$$\boxed{\chi(P) = \chi(P;t) = \sum_{x \in P} \mu(x)\, t^{n - rk\, x}}$$

↳ this uses the one-variable form of the Möbius function

Now, we will compute the characteristic polynomials for some of our standard example posets

We have the following characteristic polynomials
a) For C_n: $\chi(C_n) = t^{n-1}(t-1)$

b) For B_n: $\chi(B_n) = (t-1)^n$

c) If n has prime factorization $n = p_1^{m_1} \cdots p_k^{m_k}$ and $m = \sum_i m_i$
then $\chi(D_n) = t^{m-k}(t-1)^k$

For matroids:
The characteristic polynomial $\chi(M,t)$ of a matroid M is $\chi(\mathcal{L}(M), t)$ the characteristic polynomial of the corresponding geometric lattice

Now, we are going to move to chromatic polynomial (which is not the same as the characteristic polynomial)

Theorem 6.2: Given a graph G with chromatic polynomial $C(G,t)$ and let G_M be the corresponding matroid. Then, $C(G,t) = t^c \chi(G_M, t)$
where c is the number of connected components of G

EXAMPLE:

Going back to the group:

= the set B_3

$$C(G_1, t) = t(t-1)(t-1)(t-1) = \boxed{t(t-1)^3}$$

We know $\chi(B_3, t) = (t-1)^3$

$c = 1 \Rightarrow$ through the theorem $C(G,t) = t^c \chi(G_M, t)$

We have verified the equation for the chromatic polynomial of the example

In general to go from a matroid to a geometric lattice:

① Draw entire Boolean alg B_n

② Erase all subsets which are not flats

③ connect remaining edges

The Heron–Rota–Welsh unimodality conjecture ([32, 56, 64]) asserts that the coeffi- cients of the characteristic polynomial of a matroid form a log-concave sequence. This implies that the coefficients are unimodal.

A special case of the conjecture is an earlier conjecture by Read, asserting that the coefficients of the chromatic polynomial of a graph are unimodal. In 2009 June Huh used algebraic geometry to prove Read's unimodality conjecture [33] for graphs, and the more general Heron–Rota–Welsh conjecture for matroids represented over a field of characteristic 0. The case of matroids representable over a field of a non-zero characteristic and the case of general matroids remained open.

In 2010 June Huh and Eric Katz [36] found a different algebraic-geometric approach and proved the case of matroids repre- sentable over a field of an arbitrary characteristic. Finally, in 2015 the Heron–Rota–Welsh conjecture was proved in full generality by Karim Adiprasito, June Huh, and Eric Katz [1]. For this purpose it was necessary to extend theorems from algebraic geometry (primarily the Hodge–Riemann relations and the hard Lefschetz theorem) to cases well beyond the scope of algebraic geometry. Huh and his coauthors developed an entirely novel theory of great interest and importance.

9) June Huh's Life Story

1) **Childhood:** Huh was born in Stanford, CA when his parents were graduate students but ~~?~~ grew up in Seoul, South Korea. He was convinced that he was bad at math after receiving a poor score on an elementary school.

- He dropped out of high school to pursue poetry
- in university he studied physics, and in his sixth year he took a class by Fields medal mathematician Heisuke Hironaka

Nine years later, at the age of 34, Huh is at the pinnacle of the math world. He is best known for his proof, with the mathematicians Eric Katz and Karim Adiprasito, of a long-standing problem called the Rota conjecture.

The mathematician Gian-Carlo Rota developed a number of different conjectures that bear his name.
Even more remarkable than the proof itself is the manner in which Huh and his collaborators achieved it — by finding a way to reinterpret ideas from one area of mathematics in another where they didn't seem to belong. This past spring IAS offered Huh a long-term fellowship, a position that has been extended to only three young mathematicians before. Two of them (Vladimir Voevodsky and Ngô Bảo Châu) went on to win the Fields Medal, the highest honor in mathematics.

That Huh would achieve this status after starting mathematics so late is almost as improbable as if he had picked up a tennis racket at 18 and won Wimbledon at 20. It's the kind of out-of-nowhere journey that simply doesn't happen in mathematics today, where it usually takes years of specialized training even to be in a position to make new discoveries. Yet it would be a mistake to see Huh's breakthroughs as having come in spite of his unorthodox beginning. In many ways they're a product of his unique history — a direct result of his chance encounter, in his last year of college, with a legendary mathematician who somehow recognized a gift in Huh that Huh had never perceived himself.