Notes on the Robinson-Schensted-Knuth (RSK) Algorithm and Cauchy's Formula

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March 30, 2023

1 The Robinson-Schensted (RS) Algorithm

Recall the RS Algorithm which, given a permuation $\pi \in S_n$, outputs a pair of standard Young tableaux (SYT) P and Q. P and Q both have shape λ, where λ is some partition of n. Write out π as a $2 \times n$ matrix:

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}$$

RS begins with two empty SYTs, denoted P_0 and Q_0. The algorithm inserts the bottom row into P_0 left-to-right, and the final output is P. As $\pi(i)$ is inserted into P_{i-1} to get P_i, the number i is inserted into Q_{i-1} to get Q_i. These methods of insertion are different between P and Q.

- Insertion into P_{i-1}. The number $k = \pi(i)$ is inserted using a slide rule. Beginning at the first row, we look for an existing entry which has value greater than k. If no such entry exists, then k is appended to that row and we are done. Otherwise, suppose we find an entry $j > k$. Then we replace the entry j with value k, and "slide" j into the next row. This process repeats until an element is appended to some row. Note that if the row is empty, the input is automatically added to create a new row with length one.

- Insertion into Q_{i-1}. We note the last position in which an element was placed in P_{i-1} to get to P_i. We place i in that position in Q_{i-1} to get Q_i.

Example. Let

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}$$

P_0 and P_1 are both empty tableaux. We insert 4 into P_0 and 1 into Q_0, then continue.

$$P_1 = \begin{array}{|c|}\hline 4 \\ \hline \end{array} \qquad Q_1 = \begin{array}{|c|}\hline 1 \\ \hline \end{array}$$

$$P_2 = \begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline \end{array} \qquad Q_2 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$$

$$P_3 = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \qquad Q_3 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

$$P_4 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \qquad Q_4 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$$

Remarks: First, for P and Q each step preserves the fact that rows are increasing left-to-right, and increasing top-to-bottom. The shape of P and Q are always identical at each step. As a result, RS always outputs a pair of SYT with the same shape. Second, this algorithm is reversible. Without going into too much detail, you can imagine using entries in Q to find a position in P, and "slide backwards" to pull out an element which would have been placed at that time step. This builds up a permutation last-to-first entry. Thus,

Theorem The RS algorithm is a bijection between permutations over n elements and pairs of SYT with the same shape $\lambda \vdash n$.

Corollary $\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$, where f^λ denotes the number of SYT of shape λ.

Pf. Count both sides of the bijection. □

But π did not have to be a permutation in order for RS to produce something meaningful. Perhaps we can loosen the structure of π so that the output of this procedure is a pair of semistandard Young tableaux? We can also generalize the input, from permutations to matrices. Every permutation is encoded by some permutation matrix. This is the motivation behind RSK.

2 Generalized Permutations

Definition. Let $A = (a_{ij})_{i,j \geq 1}$ be an $\mathbb{N}$-matrix with finite support, i.e. there are only a finite number of non-zero entries in A. The *generalized permutation associated with* A is a $2 \times n$ matrix w_A,

$$w_A = \begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix},$$

such that

1. $i_1 \leq i_2 \leq \cdots \leq i_n$ (the top row is weakly increasing),
2. If $i_r = i_s$ then $j_r = j_s$ (among all j's with the same corresponding i's, they are weakly increasing), and
3. for each pair (i, j), there are exactly a_{ij} occurences of the column $\begin{bmatrix} i \\ j \end{bmatrix}$ in w_A.

Observe that since A is assumed to have finite support, w_A must always be a finitely sized matrix. Furthermore, a regular permutation is a particular type of generalized permutation, where A is a permutation matrix.

Example Consider the $\mathbb{N}$-matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

A has finite support, so we may consider it as just the 3×3 matrix where there are non-zero entries. The generalized permutation for A is

$$w_a = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 3 & 3 & 2 & 2 & 1 & 2 \end{pmatrix}.$$

To see that this is true, note that $(2, 2)$ occurs twice in w_A, and $a_{22} = 2$. Check this for each entry of A.

3 The Robinson-Schensted-Knuth (RSK) Algorithm

The RSK algorithm is identical to RS, except that we execute the procedure on input a generalized permutation.

Example Let's simulate RSK on w_A from the last example. We start with $P_0 = \emptyset$, $Q_0 = \emptyset$.

$$P_1 = \begin{array}{|c|}\hline 1 \\ \hline \end{array} \qquad Q_1 = \begin{array}{|c|}\hline 1 \\ \hline \end{array}$$

$$P_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \end{array} \qquad Q_2 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}$$

$$P_3 = \begin{array}{|c|c|c|}\hline 1 & 3 & 3 \\ \hline \end{array} \qquad Q_3 = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array}$$

$$P_4 = \begin{array}{ccc} 1 & 2 & 3 \\ 3 & & \end{array} \qquad Q_4 = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & & \end{array}$$

$$P_5 = \begin{array}{ccc} 1 & 2 & 2 \\ 3 & 3 & \end{array} \qquad Q_5 = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & \end{array}$$

$$P_6 = \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & \\ 3 & & \end{array} \qquad Q_6 = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & \\ 3 & & \end{array}$$

$$P_7 = \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 2 & 3 & & \\ 3 & & & \end{array} \qquad Q_7 = \begin{array}{cccc} 1 & 1 & 1 & 3 \\ 2 & 2 & & \\ 3 & & & \end{array}$$

The Main RSK Theorem

1. RSK is a bijection between $\mathbb{N}$-matrices with finite support and pairs of SSYT of the same shape.

2. j occurs in P exactly $\sum_i a_{ij}$ times, and i occurs in Q exactly $\sum_j a_{ij}$ times.

Pf sketch.

- Part 1: prove that each gen. perm. maps to a pair of SSYT with the same shape.

- Part 2: show that you can reverse RSK. This is not as easy as RS, but it can be done. The dilemma is that in Q, we might insert the same value i more than once, and so when going backwards we can't tell which place to start at before sliding. The key here is to observe that equal elements are inserted left-to-right in Q. Thus, we can relabel entries in Q to $1, \ldots, n$ and reverse in the same way as RS. □

4 Cauchy's Formula

Just like for the RS correspondence, there is an equivalence arising from the RSK bijection.

Theorem (Cauchy's Formula)

$$\sum_{\lambda} s_\lambda(x) s_\lambda(y) = \prod_{j,k}^{\infty} \frac{1}{1 - x_j x_k},$$

where the sum is over all partitions λ and s_λ is the Schur polynomial

$$s_\lambda(x) = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T.$$

To prove this, we first associate with each generalized permutation a polynomial over variables $x = x_1, x_2, \ldots$ and $y = y_1, y_2, \ldots$. Then we show that both sides are equal to the generating function for generalized permutations.

Let $\pi = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}$ be a generalized permutation. Let

$$\begin{aligned} \mathrm{topwt}(\pi) &= x_{a_1} x_{a_2} \cdots x_{a_n}, \\ \mathrm{bottomwt}(\pi) &= y_{b_1} y_{b_2} \cdots y_{b_n}. \end{aligned}$$

Lemma 1 $\sum_\lambda s_\lambda(x) s_\lambda(y) = \sum_\pi \mathrm{topwt}(\pi)\mathrm{bottomwt}(\pi)$, where the sum is over all generalized permutations π.

Pf. Using the definition of Schur polynomials,

$$\sum_{\lambda} s_\lambda(x) s_\lambda(y) = \sum_{\lambda} \Big(\sum_{Q \in \mathrm{SSYT}(\lambda)} x^Q \Big) \Big(\sum_{P \in \mathrm{SSYT}(\lambda)} y^P \Big) = \sum_{\lambda} \sum_{P,Q \in \mathrm{SSYT}(\lambda)} x^Q y^P.$$

Let α and β be contents for SSYT of the same shape. The coefficient of $x^\alpha y^\beta$ above is the number of pairs (P, Q) for which P and Q are SSYT with shape λ and the content of Q and P is α and β, resp.

Now observe that if $\pi \xrightarrow{\mathrm{RSK}} (P, Q)$ then $\mathrm{topwt}(\pi) = x^Q$ and $\mathrm{bottomwt}(\pi) = x^P$ exactly. The coefficient of $x^\alpha y^\beta$ is the number of generalized permutations π where $\mathrm{topwt}(\pi) = x^Q$, $\mathrm{bottomwt}(\pi) = x^P$. By the bijectivity of RSK, the coefficients for $x^\alpha y^\beta$ are equal and so we are done. □

Lemma 2 $\sum_\pi \mathrm{topwt}(\pi)\mathrm{bottomwt}(\pi) = \prod_{j,k}^{\infty} (1 - x_j y_k)^{-1}$.

Pf. Every generalized permutation can be constructed by choosing any number of each column $\begin{bmatrix} j \\ k \end{bmatrix}$ and then arranging them in order such that the three properties are preserved. For example, we can choose 3 of $\begin{bmatrix}1\\1\end{bmatrix}$, 2 of $\begin{bmatrix}2\\1\end{bmatrix}$, and 1 of $\begin{bmatrix}2\\2\end{bmatrix}$ to get the generalized permutation

$$\begin{pmatrix}1 & 1 & 1 & 2 & 2 & 2\\ 1 & 1 & 1 & 1 & 1 & 2\end{pmatrix}.$$

In fact, once we choose the columns then there is only one unique way to order them and obtain a generalized permutation. First, order the columns by non-decreasing top row. Then for each fixed top row value, order the columns by non-decreasing bottom row.

In the example above, the expression topwt(π)bottomwt(π) is equal to $(x_1y_1)^3 \cdot (x_2y_1)^2 \cdot (x_2y_2)^1$. If we consider this expression inside the sum over all generalized permutations, we "chose" the monomials $(x_1y_1)^3$, $(x_2y_1)^2$, $(x_2,y_2)^1$ (and $(x_jy_k)^0$ for all other columns) and multiplied them together. The sum is thus the sum over all ways to choose these monomials and take their product. Using the series $\sum_{n=0}^{\infty} x^n = (1-x)^{-1}$, we get

$$\begin{aligned}\sum_{\pi} \text{topwt}(\pi)\text{bottomwt}(\pi) &= \prod_{j,k}((x_jy_k)^0 + (x_jy_k)^1 + (x_jy_k)^2 + \cdots) \\ &= \prod_{j,k}^{\infty} \frac{1}{1-x_jy_k}.\end{aligned}$$

□

Cauchy's formula follows immediately.

5 RSK Symmetry and Other Corollaries

Theorem (No proof.) Let A be a $\mathbb{N}$-matrix with finite support. If $A \xrightarrow{\text{RSK}} (P,Q)$, then $A^\top \xrightarrow{\text{RSK}} (Q,P)$.

Corollary A is symmetric iff $P=Q$. *Pf.* If $P=Q$ then both A and $A^\top$ are mapped to (P,P). By bijectivity of RSK, this must mean that $A=A^\top$. On the other hand, if $A=A^\top$ then they must map to the same pair of tableaux. Then $P=Q$. □

Corollary Let $A=A^\top$ and $A \xrightarrow{\text{RSK}} (P,P)$. The map $A \mapsto P$ is a bijection between symmetric $\mathbb{N}$-matrixes and SSYT, such that the sequence of sum of the rows of A is the content of P.

Corollary

$$\sum_{\lambda \vdash n} f^\lambda = \#\{w \in S_n \mid w^2 = \mathrm{id}\}.$$

Pf. Let $w \in S_n$ and $w \stackrel{\mathrm{RSK}}{\longrightarrow} (P, Q)$, where P and Q are SYT of shape $\lambda \vdash n$. We know that w is the generalized permutation associated with a permutation matrix P_w. It is a fact that for all permutation matrices, the transpose of the matrix is its inverse. If w is an involution, then P_w is its own inverse. Then $P_w^\top = P_w$. Vice versa, if $P_w^\top = P_w$ then w is an involution. Each permutation matrix when passed through RS leads to a pair of SYT. By bijectivity then, the number of involutions in S_n is exactly the number of pairs of SYT (P, P) of shape $\lambda \vdash n$. □