Jacobi-Trudi Identities

Theorem 6 2 First J T I

for any partition λ and any $k \geq \ell(\lambda)$,

$$s_\lambda = \det(h_{\lambda_i + j - i})_{1 \leq i, j \leq k}$$

where $h_n = 0$ for all $n < 0$

expresses schur function s_λ in terms of CHSF(h_M)

Try $s_{322} = \det \begin{bmatrix} h_3 & h_4 & h_5 \\ h_1 & h_2 & h_3 \\ h_0 & h_1 & h_2 \end{bmatrix}$

Examples

$s_{22} = h_2 h_2 - h_3 h_1 = \det \begin{bmatrix} h_2 & h_3 \\ h_1 & h_2 \end{bmatrix}$

$s_{32} = \det \begin{bmatrix} h_3 & h_4 \\ h_1 & h_2 \end{bmatrix}$

$s_{211} = \det \begin{bmatrix} h_2 & h_3 & h_4 \\ h_0 & h_1 & h_2 \\ h_{-1} & h_0 & h_1 \end{bmatrix}$

Theorem 6 10 second J T I

for any partition λ and any $k \geq \lambda$

$$s_\lambda = \det(e_{\lambda'_i + j - i})_{1 \leq i, j \leq k}$$

expresses schur function s_λ in terms of ESF(e_M)

Try $s_{32} = \det \begin{bmatrix} e_3 & e_4 \\ e_1 & e_2 \end{bmatrix}$

$s_{21} = e_2 e_1 - e_3 = \det \begin{bmatrix} e_2 & e_3 \\ e_0 & e_1 \end{bmatrix}$

$s_{22} = e_2 e_2 - e_3 e_1 = \det \begin{bmatrix} e_2 & e_3 \\ e_1 & e_2 \end{bmatrix}$

Prop 4 2

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition into at most n parts,

$$s_\lambda(x_1, \ldots, x_n) = \det(h_{\lambda_i + j - i})_{i,j=1}^{k}$$

Proof

Using $\prod_{j=1}^{n}(1 + x_j t) = \sum_{i=0}^{\infty} e_i(x_1, x_2 \ldots x_n) t^i$ => generating fn for ESF

$$\sum_{i=0}^{\infty} h_i(x_1, x_2 \ldots x_n) t^i = \prod_{j=1}^{n} \frac{1}{1 - x_j t}$$ => generating fn for CSF

Let $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_k)$ be any sequence of non-negative integers

Define $k \times k$ matrices $A_\alpha = (x_j^{\alpha_i})$ $H_\alpha = (h_{\alpha_i - k + j})$ $M = ((-1)^{k-i} e_{k-i}^{(j)})$

$e_{k-i}^{(j)}$ is $(k-i)$th ESF in all variables except x_j

Prove $\det(H_\alpha) = \dfrac{\det(A_\alpha)}{\det(M)}$, $A_\alpha = H_\alpha M$

$$\sum_{m=0}^{\infty} h_m t^m \sum_{n=0}^{k-1} e_n^{(\ell)} (-t)^n = \prod_{m=1}^{k} \frac{1}{1 - x_m t} \prod_{\substack{n=1 \\ n \neq 1}}^{k} (1 - x_n t)$$

$$= \frac{1}{1 - x_1 t} = 1 + x_1 t + x_1^2 t^2$$

$$\sum_{j=1}^{k} h_{a_i - k + j} (-1)^{k-j} e_{k-j}^{(\ell)} = x_\ell^{a_i} \Rightarrow H_\alpha M = A_\alpha$$

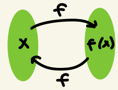
when $\alpha = (k-1, k-2, \ldots 0)$ H_α has 1's on main diagonal

↳ det = 1

The involution ω

replace e's and h's, s_λ and $s_{\lambda'}$

isomorphism from Λ to Λ e_λ to h_λ, h_λ to e_λ, s_λ to $s_{\lambda'}$



Prop 6.16 following are equivalent for any linear transf $\omega\ \Lambda \to \Lambda$

i) $\omega(e_\lambda) = h_\lambda$

ii) $\omega(h_\lambda) = e_\lambda$

iii) $\omega(s_\lambda) = s_{\lambda'}$

there is unique lin transf $\omega\ \Lambda \to \Lambda$ which satisfies (i)-(iii), ω is involution

Proof

ESF form basis for Λ, there is unique lin trans $\omega_e\ \Lambda \to \Lambda$ that satisfies (i)

$$\omega_h(s_\lambda) = \omega_h(\det(h_{\lambda_i + j - i})_{1 \le i, j \le k})$$
$$= \det(e_{\lambda_i + j - i})_{1 \le i, j \le k}$$
$$= s_{\lambda'}$$

Therefore ω_h satisfies (iii) so $\omega_h = \omega_s$, $\omega_e = \omega_s$, $\omega_e = \omega_s = \omega_h$

$\hookrightarrow$ shows involution since $\lambda'' = \lambda$ for $\forall \vdash \lambda$

Definition suppose λ is a partition w/ a_k parts of size k, μ is partition w/ b_k parts of size k

Then $\lambda \cup \mu$ denotes partition with $a_k + b_k$ parts of size k for $\forall k$

$$e_\lambda e_\mu = e_{\lambda \cup \mu}, \quad h_\lambda h_\mu = h_{\lambda \cup \mu}$$

Prop 6.18

for any symmetric functions f and g

$\omega(fg) = \omega(f)\,\omega(g)$

Proof $f = \sum_\lambda a_\lambda e_\lambda$ $g = \sum_\mu b_\mu e_\mu$

LHS $fg = \sum_{\lambda,\mu} a_\lambda b_\mu e_\lambda e_\mu$

so $\omega(fg) = \sum_{\lambda,\mu} a_\lambda b_\mu h_{\lambda \cup \mu}$

RHS $\omega(f)\omega(g) = \sum_{\lambda,\mu} a_\lambda b_\mu\, \omega(e_\lambda)\, \omega(e_\mu)$
$= \sum_{\lambda,\mu} a_\lambda b_\mu h_\lambda h_\mu$
$= \sum_{\lambda,\mu} a_\lambda b_\mu h_{\lambda \cup \mu}$

from this, $\omega(p_1) = p_1$, $\omega(p_2) = -p_2$

Prop 6.19

for all $n \ge 1$, $\omega(p_n) = (-1)^{n-1} p_n$

Equation 34

for all $k \ge 1$, $p_k = \sum_{j=1}^{k} (-1)^{j-1} j\, e_j h_{k-j}$

Proof of Equation 3 4.

$$p_k = \sum_{j=1}^{k} (-1)^{j-1} j e_j h_{k-j}$$

$\boxed{1}\boxed{3}\boxed{4^*}\boxed{7}$ — $x_1 x_3 x_4 x_7$ $\quad$ $\boxed{2}\boxed{2}\boxed{4}$ — $x_2^2 x_4$

$(1 \times j)$ strict row $\quad$ $1 \times (k-j)$ weak row

1) $EH_\ell = \{ \square\square\square\square, \square\square\square\square \}$ Let $EH = \bigcup_{\ell=1}^{k} EH_\ell$

2) Define $f: EH \Rightarrow EH$, r_{mark} = entry marked

r = smallest # in either tile $\neq r_{mark}$

$$f(L_T, R_T) = \begin{cases} (L_T \cup \{r\}, R_T \setminus \{r\}) & \text{if } r \notin L_T \\ (L_T \setminus \{r\}, R_T \cup \{r\}) & \text{if } r \in L_T \\ (L_T, R_T) & \text{if } r \text{ does not exist} \end{cases}$$

$$\sum_{j=1}^{k} (-1)^{j-1} j\, e_j h_{k-j} = \sum_{j=1}^{k} \sum_{T \in EH_j} (-1)^{j-1} x^{\vec{T}}$$

Proof 6 19

for all $n \geq 1$, $\omega(p_n) = (-1)^{n-1} p_n$

apply ω to (3 4) $\rightsquigarrow$ for all $k \geq 1$

$$\omega(p_n) = \sum_{j=1}^{n} (-1)^{j-1} \omega(e_j h_{k-j}) \quad p_k = \sum_{j=1}^{k} (-1)^{j-1} j e_j h_{k-j}$$

$$= \sum_{j=1}^{n} (-1)^{j-1} j\, \omega(e_j)\, \omega(h_{k-j})$$

$$= \sum_{j=1}^{n} (-1)^{j-1} j\, h_j e_{k-j}$$

$$= (-1)^{n-1} p_n$$

Prop 6 20

for any partition λ,

$\omega(p_\lambda) = (-1)^{|\lambda| - \ell(\lambda)} p_\lambda$

Proof $\omega(p_\lambda) = \omega\left(\prod_{j=1}^{\ell(\lambda)} p_{\lambda_j}\right) = \prod_{j=1}^{\ell(\lambda)} \omega(p_{\lambda_j})$

$$= \prod_{j=1}^{\ell(\lambda)} (-1)^{\lambda_j - 1} p_{\lambda_j} \Rightarrow (-1)^{|\lambda| - \ell(\lambda)} p_\lambda$$

Apply Involution ω to JTI

$$\omega(s_\lambda) = \omega(\det(h_{\lambda_i + j - i}))$$

$$s_{\lambda'} = \det(e_{\lambda_i + j - i})$$