Lecture on Stirling Numbers by Doran Sekaran

Review

- Def: $\forall n, k \in \mathbb{Z}^+$, the elementary symmetric polynomial (ESP) $e_k(X_n)$ is given by $e_k(X_n) = \sum_{\substack{J \subseteq [n] \\ |J| = k}} \prod_{j \in J} X_j$
- Example: $e_2(X_3) = X_1X_2 + X_1X_3 + X_2X_3$
- Def: $\forall n, k \in \mathbb{Z}^+$, the complete homogeneous symmetric polynomial (CHSP) $h_k(X_n)$ is given by $h_k(X_n) = \sum_{\substack{J \subseteq [[n]] \\ |J| = k}} \prod_{j \in J} X_j$
- Example: $h_2(X_3) = X_1^2 + X_2^2 + X_3^2 + X_1X_2 + X_1X_3 + X_2X_3$
- Note: $[n]$ is the set $\{1, 2 \ldots n\}$, $[[n]]$ is the same set but with infinite copies of each element

Symmetric Function Identities

- Proposition 1: $\forall k \geq 0$, $\forall n \geq 1$, we have

  (1.1) $e_k(X_n) = e_k(X_{n-1}) + X_n e_{k-1}(X_{n-1})$ and

  (1.2) $h_k(X_n) = h_k(X_{n-1}) + X_n h_{k-1}(X_n)$.
- Proof of 1.1:

$$e_k(X_n) = \sum_{\substack{J \subseteq [n] \\ |J| = k}} \prod_{j \in J} X_j = \underbrace{\sum_{\substack{J \subseteq [n-1] \\ |J| = k}} \prod_{j \in J} X_j}_{\text{Terms without factor } X_n} + \underbrace{\sum_{\substack{J \subseteq [n-1] \\ |J| = k-1}} X_n \prod_{j \in J} X_j}_{\text{Terms with factor } X_n}$$

$$\Rightarrow e_k(X_n) = e_k(X_{n-1}) + X_n e_{k-1}(X_{n-1})$$

- Example: $e_2(X_3) = X_1X_2 + X_1X_3 + X_2X_3 = X_1X_2 + X_3(X_1 + X_2) = e_2(X_2) + X_3 e_1(X_2)$
- Proof of 1.2:

$$h_k(X_n) = \sum_{\substack{J \subseteq [[n]] \\ |J| = k}} \prod_{j \in J} X_j = \underbrace{\sum_{\substack{J \subseteq [[n-1]] \\ |J| = k}} \prod_{j \in J} X_j}_{\text{Terms without factor } X_n} + \underbrace{\sum_{\substack{J \subseteq [[n]] \\ |J| = k-1}} X_n \prod_{j \in J} X_j}_{\text{Terms with factor } X_n}$$

$$\Rightarrow h_k(X_n) = h_k(X_{n-1}) + X_n e_{k-1}(X_n)$$

- Example: $h_2(X_3) = X_1^2 + X_2^2 + X_3^2 + X_1X_2 + X_1X_3 + X_2X_3$
  $= X_1^2 + X_2^2 + X_1X_2 + X_3(X_1 + X_2 + X_3) = h_2(X_2) + X_3 h_1(X_3)$
- Note: There's a subtle but fundamental difference between eq. 1.1 and 1.2

Binomial Coefficients

- Def: The binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the number of ways of picking k unordered outcomes from n possibilities
- Proposition 2: $\forall k \geq 0, \forall n \geq 1$, we have

  (2.1) $e_k(\overbrace{1,\ldots,1}^{n \text{ times}}) = \binom{n}{k}$ and

  (2.2) $h_k(\overbrace{1,\ldots,1}^{n \text{ times}}) = \binom{n+k-1}{k}$
- Proof of 2.1:

  $e_k(\overbrace{1,\ldots,1}^{n \text{ times}})$ is the number of terms in $e_k(X_n)$. Each term contains k variables (each raised to the first power) chosen from a set of n variables. Thus, there are $\binom{n}{k}$ terms. Thus, $e_k(\overbrace{1,\ldots,1}^{n \text{ times}}) = \binom{n}{k}$
- Proof of 2.2:

  $h_k(\overbrace{1,\ldots,1}^{n \text{ times}})$ is the number of terms in $h_k(X_n)$. Each term can be represented as a string of "Stars and bars", where stars represent exponents and bars determine how the exponents are distributed amongst variables. For example, $X_1 X_3^3 X_4^2$, a term in $h_6(X_4)$ is represented by $* \mid \mid * * * \mid * *$

  For terms in $h_k(X_n)$, there are always k stars and $n-1$ bars. We can choose k positions out of the $n+k-1$ positions for the stars. Thus, there are $\binom{n+k-1}{k}$ total strings, or terms. Thus, $h_k(\overbrace{1,\ldots,1}^{n \text{ times}}) = \binom{n+k-1}{k}$

Stirling Numbers

- Def: $\forall n,k \in \mathbb{Z}^+$, the Stirling number of the first kind $\left[{n \atop k}\right]$ is the number of permutations of $[n]$ with exactly k cycles.
- Note: The permutations of $[n]$ are the elements of S_n
- Example: $\left[{3 \atop 1}\right]=2$. $S_3=\{(1)(2)(3),(12)(3),(13)(2),(23)(1),(123),(132)\}$
2 of the 6 elements are composed of one cycle.
- Def: For any set A, a set partition of A is a set $\{B_1,\dots,B_k\}$ of non-empty subsets of A s.t. $A=\bigcup_{j=1}^{k} B_j$ and if $j \neq \ell$, then $B_j \cap B_\ell = \emptyset$. The set B_j are the blocks of the partition
- Def: $\forall n,k \in \mathbb{Z}^+$, the Stirling number of the Second Kind $\left\{{n \atop k}\right\}$ is the number of set partitions of $[n]$ with exactly k blocks
- Note: The order of blocks and elements within blocks does not matter.
- Example: $\left\{{3 \atop 1}\right\}=1$. (123)
- Proposition 3.1: $\forall n \geq 1$, we have $\left[{n \atop 0}\right]=0$ and $\left[{n \atop n}\right]=1$ and $\forall n \geq 2$ and $1 \leq k \leq n-1$, we have $\left[{n \atop k}\right]=\left[{n-1 \atop k-1}\right]+(n-1)\left[{n-1 \atop k}\right]$
- Proof of 3.1: : There are no elements of S_n with 0 cycles and there is one element of S_n with n cycles (the identity).

Take all k-cycle elements of S_n where n is in a cycle alone.
Removing n, we have all the elements of S_{n-1} with exactly $k-1$ cycles
There are $\left[{n-1 \atop k-1}\right]$ such elements (e.g. $(12)(34)(5)$)
Take all k-cycle elements of S_n where n is not in a cycle alone
Removing n, we have all the elements of S_{n-1} w/ exactly k cycles
But there are $n-1$ elements in S_n that lead to the same element in S_{n-1}
(e.g. $(12)(345)$, $(12)(354)$, $(125)(34)$, $(152)(34)$ all lead to $(12)(34)$)
Thus, there are $(n-1)\left[{n-1 \atop k}\right]$ k-cycle elements of S_n where n is not in a cycle alone.
Thus, the number of k-cycle elements of S_n $\left[{n \atop k}\right]=\left[{n-1 \atop k-1}\right]+(n-1)\left[{n-1 \atop k}\right]$

·Proposition 3.2: $\forall n \geq 1$, we have $\genfrac{\{}{\}}{0pt}{}{n}{1} = \genfrac{\{}{\}}{0pt}{}{n}{n} = 1$
and $\forall n \geq 2$ and $2 \leq k \leq n-1$ we have $\genfrac{\{}{\}}{0pt}{}{n}{k} = \genfrac{\{}{\}}{0pt}{}{n-1}{k-1} + k\genfrac{\{}{\}}{0pt}{}{n-1}{k}$

·Proof of 3.2: There is one way to arrange n elements in 1 block and one way to arrange n elements in n blocks.
Take all k-block arrangements of n elements where n is in a block alone
Removing n, we have all arrangements of $n-1$ elements w/ exactly $k-1$ blocks
There are $\genfrac{\{}{\}}{0pt}{}{n-1}{k-1}$ such arrangements (e.g. (12)(34)(5))
Take all k-block arrangements of n elements where n is not in a block alone
Removing n, we have all arrangements of $n-1$ elements w/ exactly k blocks.
But there are k arrangements of n elements that lead to the same arrangement of $n-1$ elements (e.g. (12)(345), (125)(34) both lead to (12)(34))
Thus, there are $k\genfrac{\{}{\}}{0pt}{}{n-1}{k}$ k-block arrangements of n elements where n is not in a block alone. Thus, the number of k-block arrangements of n elements $\genfrac{\{}{\}}{0pt}{}{n}{k} = \genfrac{\{}{\}}{0pt}{}{n-1}{k-1} + k\genfrac{\{}{\}}{0pt}{}{n-1}{k}$

·Proposition 4.1: $\forall n \geq 1$ and $0 \leq k \leq n$, we have
$$e_k(1,2,\dots n) = \genfrac{[}{]}{0pt}{}{n+1}{n+1-k}$$

·Proof of prop 4.1: Proceed by induction on n
Base case: $n=1 \Rightarrow k=0$ or 1
$e_0(1) = 1 = \genfrac{[}{]}{0pt}{}{2}{2} = \genfrac{[}{]}{0pt}{}{1+1}{1+1-0}$, in general $e_0(1,2\dots n) = 1 = \genfrac{[}{]}{0pt}{}{n+1}{n+1} = \genfrac{[}{]}{0pt}{}{n+1}{n+1-0}$
$e_1(1) = 1 = \genfrac{[}{]}{0pt}{}{2}{1} = \genfrac{[}{]}{0pt}{}{1+1}{1+1-1}$, in general $e_n(1,2\dots n) = n! = \genfrac{[}{]}{0pt}{}{n+1}{1} = \genfrac{[}{]}{0pt}{}{n+1}{n+1-n}$
Induction step: Suppose $n \geq 2$ and the statement holds for $n-1$
That is $e_k(1,2,\dots n-1) = \genfrac{[}{]}{0pt}{}{n}{n-k}$ and $e_{k-1}(1,2,\dots n-1) = \genfrac{[}{]}{0pt}{}{n}{n-k+1}$
Recall prop 1.1 $e_k(X_n) = e_k(X_{n-1}) + x_n e_{k-1}(X_{n-1})$
$\Rightarrow e_k(1,2\dots,n) = e_k(1,2,\dots n-1) + n\, e_{k-1}(1,2,\dots n-1)$
$\Rightarrow e_k(1,2\dots,n) = \genfrac{[}{]}{0pt}{}{n}{n-k} + n\genfrac{[}{]}{0pt}{}{n}{n-k+1}$
Recall prop 3.1 $\genfrac{[}{]}{0pt}{}{n}{k} = \genfrac{[}{]}{0pt}{}{n-1}{k-1} + (n-1)\genfrac{[}{]}{0pt}{}{n-1}{k}$
$\Rightarrow \genfrac{[}{]}{0pt}{}{n+1}{n-k+1} = \genfrac{[}{]}{0pt}{}{n}{n-k} + n\genfrac{[}{]}{0pt}{}{n}{n-k+1}$
$\Rightarrow e_k(1,2,\dots n) = \genfrac{[}{]}{0pt}{}{n+1}{n+1-k}$

· Proposition 4.2: $\forall n \geq 1$ and $0 \leq k \leq n$, we have
$h_k(1,2,\dots,n) = \left\{ {n+k \atop n} \right\}$

· Proof of prop 4.2: Proceed by induction on $n+k$

Base case: $n+k=1 \Rightarrow n=1, k=0$ $h_0(1) = 1 = \left\{ {1 \atop 1} \right\} = \left\{ {1+0 \atop 1} \right\}$

Induction step: Suppose $n+k = N \geq 2$ and the statement holds $\forall$ n,k pairs s.t. $n+k = N-1$

That is $h_k(1,2,\dots,n-1) = \left\{ {n-1+k \atop n-1} \right\}$ and $h_{k-1}(1,2,\dots,n) = \left\{ {n+k-1 \atop n} \right\}$

Recall prop 1.2 $h_k(X_n) = h_k(X_{n-1}) + X_n h_{k-1}(X_n)$

$\Rightarrow h_k(1,2,\dots n) = h_k(1,2,\dots n-1) + n\, h_{k-1}(1,2,\dots n)$

$\Rightarrow h_k(1,2,\dots n) = \left\{ {n-1+k \atop n-1} \right\} + n \left\{ {n+k-1 \atop n} \right\}$

Recall prop 3.2 $\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$

$\Rightarrow \left\{ {n+k \atop n} \right\} = \left\{ {n+k-1 \atop n-1} \right\} + n \left\{ {n+k-1 \atop n} \right\}$

$\Rightarrow h_k(1,2,\dots n) = \left\{ {n+k \atop n} \right\}$

We can repeat this procedure to get this formula $\forall$ n,k pairs s.t. $n+k=N$

Combinatorial Identities

Proposition 5: $\forall n \geq 1$ $\sum_{j=0}^{n} (-1)^j e_j h_{n-j} = 0$

Proof given in previous lecture. Remember e_k and h_k are symmetric functions (symmetric polynomials with infinite variables).

· Evaluating them at specific points, however, they behave like symmetric polynomials.

Proposition 6: $\forall m,n \geq 1$, we have
$$\sum_{j=0}^{m} (-1)^j \binom{n}{j} \binom{m+n-j-1}{m-j} = 0$$

Proof of proposition 6: From proposition 5 $\sum_{j=0}^{m} (-1)^j e_j h_{m-j} = 0$

Set $X_r = 1$ $\forall r$ $1 \leq r \leq n$, $X_r = 0$ $\forall r > n$

$\sum_{j=0}^{m} (-1)^j e_j(\underbrace{1,\dots,1}_{n \text{ times}})\, h_{m-j}(\underbrace{1,\dots,1}_{n \text{ times}}) = 0$

$\Rightarrow$ (from prop 2.1 and 2.2) $\sum_{j=0}^{m} (-1)^j \binom{n}{j} \binom{n+m-j-1}{m-j} = 0$

Proposition 7: $\forall m,n \geq 1$, we have
$$\sum_{j=0}^{m} (-1)^j \left[{n+1 \atop n+1-j} \right] \left\{ {n+m-j \atop n} \right\} = 0$$

Proof of proposition 7: From proposition 5 $\sum_{j=0}^{m} (-1)^j e_j h_{m-j} = 0$

Set $X_r = r$ $\forall r$ $1 \leq r \leq n$, $X_r = 0$ $\forall r > n$

$\sum_{j=0}^{m} (-1)^j e_j(1,2\dots,n)\, h_{m-j}(1,2\dots n) = 0$

$\Rightarrow$ (from prop 4.1 and 4.2) $\sum_{j=0}^{m} (-1)^j \left[{n+1 \atop n+1-j} \right] \left\{ {n+m-j \atop n} \right\} = 0$